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Note on: Lot-sizing for inventory systems with product recovery

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Abstract

In this article we revisit the paper by Teunter (2004), appeared in Computers and Industrial Engineering. For this model Teunter proposed an approach leading to an approximate solution. Here we propose an optimization procedure, which leads to policies with integer set up numbers in the production and the remanufacturing facilities i.e. to the optimal policy.

Keywords: Product returns; Inventory; Recovery; EOQ

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1. Introduction

Teunter (2004) presented an inventory model, where the stationary demand is satisfied though two modes. One is by new ordered/produced products and the other by recovered used products which recovery brings back to 'as good as new' condition. All relevant costs i.e. ordering/production and recovery set up, holding new/recovered items, holding recoverable items are constant. He considers policies that alternate one production lot with a fixed number *R* of recovery lots respectively, in short (1, *R*) policies and one recovery lot with a fixed number *P* of production lots, in short (*P*, 1) policies. In this class of policies Teunter derived simple closed type formulas for the optimal procurement/production and recovery lot sizes. These formulas are more general than the ones given in Nahmias and Rivera (1979) and Koh, Hwang, Sohn and Ko (2002) as they are valid for infinite and finite recovery and production rates respectively.

The approach used in his analysis is to minimize the total cost function $TC(Q_p, Q_r)$, w.r.t. to the procurement/production Q_p and recovery Q_r lot sizes, treating them as continuous variables. Treating lot sizes as continuous variables, in cases where these have to be integer, is common practice in inventory control literature and really it does not create any problem. He then obtained *R*, *P* using equations connected the above variables. The so obtained values are truncated, if necessary, to the nearest integer and the so obtained policy is applied. In the case of Teunter's model the obtained values of Q_p, Q_r are used to calculate *P* and *R*. If the values for *R* or *P* are not integers the policy cannot be applied. To overcome this difficulty, the author suggests suitable modifications. He first truncates the obtained values of *R*, *P* to the nearest integer, greater or equal to one. Next using these values he modifies the initially obtained values of Q_p in (1, *R*) policies and the Q_r in (*P*, 1) policies. The resulting policy can be applied and the relevant cost is calculated. In this paper we present an approach, which leads directly to the optimal policy with *R*, *P* integers. These values are then used to obtain the lot sizes Q_p and Q_r and to calculate the optimal cost.

2. Model

The notation in Teunter's (2004) model is:

d	Demand rate
f	Return fraction (return rate fd)
p	Production rate
r	Recovery rate
K_p	Ordering (setup) cost per production lot
K_r	Ordering (setup) cost per recovery lot
h_r	Holding cost per recoverable item per time unit
h_s	Holding cost per serviceable item per time unit
Q_p	Production lot size
Q_r	Recovery lot size

2.1. Policy (1,R): One manufacturing against R remanufacturing opportunities

For this class of policies, the total cost per unit of time is given by:

$$TC(Q_p, Q_r) = \frac{K_p d(1-f)}{Q_p} + \frac{K_r df}{Q_r} + \frac{h_s((1-f)(1-d/p)Q_p + f(1-d/r)Q_r)}{2} + \frac{h_r f(Q_p + (1-d/r)Q_r)}{2} \quad (1)$$

The variables Q_r , Q_p and R are connected via the relation

$$RQ_r(1-f) = Q_p f \,. \quad (2)$$

Teunter minimized $TC(Q_p, Q_r)$ w.r.t. Q_p, Q_r and using (2) he obtained R. The so obtained R is not in general integer. To make it integer, he truncates this R to the nearest integer, say $\tilde{R} = \max\{1, [R]\}$, greater or equal to one and using this truncation, he replaces the initially obtained Q_p value by the one obtained though the relation

$$\tilde{Q}_p = \frac{\tilde{R} Q_r (1-f)}{f} .$$

Here we shall approach the solution of this problem in a different way.

From (2) we have that:

$$Q_r = \frac{Q_p f}{R(1-f)}. \tag{3}$$

Replacing this Q_r into (1) the total cost per unit of time becomes:

$$TC(Q_p,R) = \frac{K_p d(1-f) + RK_r d(1-f)}{Q_p} + Q_p\left[\frac{h_s(1-f)(1-d/p)}{2} + \frac{h_r f}{2}\right] + \frac{Q_p f^2(1-d/r)(h_s+h_r)}{2R(1-f)}. \tag{4}$$

If we set

$A = K_p d(1-f) + RK_r d(1-f) = A_1 + A_2 R$, where $A_1 = K_p d(1-f) \geq 0$ and $A_2 = K_r d(1-f) \geq 0$,

$$B = \frac{h_s(1-f)(1-d/p)}{2} + \frac{h_r f}{2} \geq 0 \text{ and } C = \frac{f^2(1-d/r)(h_s+h_r)}{2R(1-f)} = \frac{C_1}{R}, C_1 = \frac{f^2(1-d/r)(h_s+h_r)}{2(1-f)} \geq 0, \tag{5}$$

we can rewrite $TC(Q_p,R)$ as:

$$TC(Q_p,R) = \frac{A}{Q_p} + Q_p(B+C). \tag{6}$$

Now the problem becomes: find the minimum of $TC(Q_p,R)$ w.r.t. R and Q_p. The approach we follow is first finding the minimum of this function w.r.t. Q_p. The minimizing point will be a function of R, say $Q_p(R)$. Replaces this into the objective function and minimize the objective w.r.t. R.

From (6) we see that $TC(Q_p,R)$ is convex in Q_p and so attains its minimum at

$$Q_p^*(R) = \sqrt{\frac{A}{B+C}} = \sqrt{\frac{A_1 + RA_2}{B + \frac{C_1}{R}}}. \tag{7}$$

Substituting $Q_p^*(R)$ into (6) yields:

$$TC(Q_p^*(R),R) = 2\sqrt{A_1 B + A_2 C_1 + A_2 BR + \frac{A_1 C_1}{R}}. \tag{8}$$

Since R is integer we use the difference function

$$\Delta TC(Q_p^*,R) = TC(Q_p^*,R) - TC(Q_p^*,R-1),\ R \geq 2$$

for the location of optimal R which in our case is:

$$\Delta TC(Q_p^*,R)=TC(Q_p^*,R)-TC(Q_p^*,R-1)=\frac{2(A_2B-\frac{A_1C_1}{R(R-1)})}{\sqrt{A_1B+A_2C_1+RA_2B+\frac{A_1C_1}{R}}+\sqrt{A_1B+A_2C_1+(R-1)A_2B+\frac{A_1C_1}{R-1}}} \quad (9)$$

From (9) we see that if $\frac{A_1C_1}{A_2B}\leq 2$, then $\Delta TC(Q_p^*,R)\geq 0$ for any $R\geq 2$ and the optimum is at $R^*=1$.

If this is not the case, then there always exists a $R^*\geq 2$ such that $\Delta TC(Q_p^*,R)<0$ for all $R\leq R^*$ and $\Delta TC(Q_p^*,R)\geq 0$ for all $R>R^*$. Simple algebra on these inequalities gives that this R^* satisfies the double inequality

$$R^*(R^*-1)<\frac{A_1C_1}{A_2B}\leq R^*(R^*+1),\quad R^*\geq 2. \quad (10)$$

In case that $R^*(R^*+1)=\frac{A_1C_1}{A_2B}$, we have two equivalent solutions and we agree to keep the smallest one.

The integer value of R^* obtained from (10), is used in (7) to calculate $Q_p^*(R^*)$ and the resulting policy can be implemented to give the cost.

We apply this approach to the example proposed by Teunter. The data of the example are:

d=1000, f=0.8, p=5000, r=3000, K_p=20, K_r=5, h_r=2 and h_s=10.

With these data we have $A_1=4000$, $A_2=1000$, $B=1.6$, $C_1=12.8$ and the cost function is:

$$TC(Q_p,R)=\frac{4000+1000R}{Q_p}+Q_p(1.6+\frac{12.8}{R}).$$

From (10) we get R^*=6 and (7) gives $Q_p^*=51.75$ and finally $TC(Q_p^*,R^*)$=386.44 . From (3) we find $Q_r^*=34.5$. The policy given by Teunter has Q_p^*=53.03, Q_r^*=35.35 and $TC(Q_p^*,R^*)$=386.55. In this example the deviations for the lot sizes and the total cost are negligible. Computational experience shows that the two approaches give quite similar results, in case that the exact R obtained using Teunter's approach is greater than one. In case that this R is smaller than one the deviations are significant. This is evident in the examples given in table 1 and suggests that in this case the approximate approach used by Teunter leads to costs much higher than the optimal.

2.2. Policy (P,1): P manufacturing opportunities against one remanufacturing

The total cost per unit time in this case is:

$$TC(Q_p, Q_r) = \frac{K_p d(1-f)}{Q_p} + \frac{K_r df}{Q_r} + \frac{h_s((1-f)(1-d/p)Q_p + f(1-d/r)Q_r)}{2} + \frac{h_r(1-fd/r)Q_r}{2} \quad (11)$$

and P is fully determined by the lot-sizes via the relation

$$Q_r(1-f) = PQ_p f. \quad (12)$$

Teunter minimized $TC(Q_p, Q_r)$ w.r.t. Q_p, Q_r and using (12) he obtained P. The so obtained P is not in general integer. To make it integer, he truncates this P to the nearest integer, say $\tilde{P} = \max\{1, [P]\}$, greater or equal to one and using this truncation, he replaces the initially obtained Q_r value by the one obtained though the relation:

$$\tilde{Q}_r = \frac{\tilde{P} Q_p f}{1-f}.$$

Here we shall approach the solution of this problem in a different way.

From (12) we have that:

$$Q_p = \frac{Q_r(1-f)}{Pf}. \quad (13)$$

So using (13), the total cost per unit (11) becomes:

$$TC(Q_r, P) = \frac{PK_p df + K_r df}{Q_r} + Q_r\left[\frac{h_s(1-f)^2(1-d/p)}{2Pf} + \frac{h_s f(1-d/r)}{2} + \frac{h_r(1-fd/r)}{2}\right]. \quad (14)$$

If we set

$A = PK_p df + K_r df = PA_1 + A_2$, where $A_1 = K_p df \geq 0$ and $A_2 = K_r df \geq 0$,

$B = \frac{h_s(1-f)^2(1-d/p)}{2Pf} + \frac{h_s f(1-d/r)}{2} + \frac{h_r(1-fd/r)}{2} = \frac{B_1}{P} + B_2$,

where $B_1 = \frac{h_s(1-f)^2(1-d/p)}{2f} \geq 0$ and $B_2 = \frac{h_s f(1-d/r)}{2} + \frac{h_r(1-fd/r)}{2} \geq 0$ (15)

we can rewrite $TC(Q_r, P)$ as:

$$TC(Q_r,P)=\frac{A}{Q_r}+Q_rB. \tag{16}$$

From (16), we see that $TC(Q_r,P)$ is convex in Q_r and attains its minimum at

$$Q_r^*(P)=\sqrt{\frac{A}{B}}=\sqrt{\frac{PA_1+A_2}{\frac{B_1}{P}+B_2}}. \tag{17}$$

Substituting (17) into (16) yields:

$$TC(Q_r^*(P),P)=2\sqrt{A_1B_1+A_2B_2+A_1B_2P+\frac{A_2B_1}{P}}. \tag{18}$$

The difference function

$$\Delta TC(Q_r^*,P)=TC(Q_r^*,P)-TC(Q_r^*,P-1),\ P\geq 2$$

of $TC(Q_r^*,R)$ shows that:

$$\Delta TC(Q_r^*,P)=TC(Q_r^*,P)-TC(Q_r^*,P-1)=\frac{2(A_1B_2-\frac{A_2B_1}{P(P-1)})}{\sqrt{A_1B_1+A_2B_2+PA_1B_2+\frac{A_2B_1}{P}}+\sqrt{A_1B_1+A_2B_2+(P-1)A_1B_2+\frac{A_2B_1}{P-1}}}. \tag{19}$$

Following the same reasoning as previously we can see that if

$$0<\frac{A_2B_1}{A_1B_2}\leq 2$$

then the optimum is at $P^*=1$. If this is not the case, then there always exists a $P^*\geq 2$ such that $\Delta TC(Q_r^*,P)<0$ for all $P\leq P^*$ and $\Delta TC(Q_r^*,P)\geq 0$ for all $P>P^*$. Simple algebra on these inequalities gives that this P^* satisfies the double inequality

$$P^*(P^*-1)<\frac{A_2B_1}{A_1B_2}\leq P^*(P^*+1),\ P^*\geq 2. \tag{20}$$

In case that $P^*(P^*+1)=\frac{A_2B_1}{A_1B_2}$, we have two equivalent solutions and we agree to always take the smallest one.

For the Teunter's example we have that: $A_1=16000$, $A_2=4000$, $B_1=0.2$ and $B_2=3.4$. The cost function is:

$$TC(Q_r,P)=\frac{16000P+4000}{Q_r}+Q_r(3.4+\frac{0.2}{P}).$$

From (20), we get $P^*=1$ and (17) gives $Q_r^*=74.54$ and finally $TC(Q_r^*,P^*)$=536.66. From (13) we find that $Q_p^*=18.63$. The policy given by Teunter has Q_p^*=70.71, Q_r^*=282.8 and $TC(Q_p^*,R^*)$=1088.9. In this example the deviations for the lot sizes and the total cost are very significant. Computational experience shows that the two approaches give quite similar results, in case that the exact *P* obtained using Teunter's approach is greater than one. In case that this *P* is smaller than one the deviations are significant. This is evident in the examples given in table 2 and suggests that in this case the approximate approach used by Teunter leads to costs much higher than the optimal.

3. Conclusion

In this paper we propose a solution method for Teunter's (2004) model which leads to integer values for the parameters *R, P* in the set of policies (1, *R)* and (*P*,1) and subsequently to the optimal policy. This is an exact approach and comparing the results obtained, to those given by Teunter's approximate method, we see that in some cases Teunter's algorithm performs very well, while in other cases the cost deviations from the optimal are significant and his method should not be applied.

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Data								Teunter's exact solution				Teunter's approximated solution				Proposed optimal solution				% Cost Deviation
d	f	p	r	K_p	K_r	h_r	h_s	Q_p	Q_r	R	$TC(Q_p,Q_r)$	R	Q_p	Q_r	$TC(Q_p,Q_r)$	R	Q_p	Q_r	$TC(Q_p,Q_r)$	Penalty[a]
1000	0,8	5000	3000	20	5	2	10	50,00	35,36	5,66	386,27	6	53,03	35,36	386,55	6	51,75	34,50	386,44	0,0003
1000	0,2	4000	2500	10	5	4	6	60,30	40,83	0,37	314,32	1	163,29	40,83	457,24	1	71,46	17,86	335,86	0,2655
500	0,3	1000	700	10	10	5	8	40,35	51,89	0,33	231,31	1	121,07	51,89	347,03	1	54,13	23,20	258,62	0,2548
500	0,3	2000	1000	20	12	10	12	38,80	33,03	0,50	469,83	1	77,06	33,03	558,19	1	45,72	19,59	489,90	0,1223
800	0,5	2400	1500	20	8	4	15	47,81	37,99	1,26	503,11	1	37,99	37,99	511,98	1	44,26	44,26	506,07	0,0115
1000	0,7	3000	2000	20	20	2	10	59,41	81,65	1,69	544,92	2	69,98	81,65	547,64	2	65,86	76,83	546,63	0,0018
20	0,8	50	35	30	20	5	6	7,13	13,03	2,19	82,79	2	6,51	13,03	82,93	2	6,76	13,51	82,87	0,0007
20	0,8	100	50	30	25	2	7	9,39	13,61	2,76	84,34	3	10,20	13,61	84,43	3	9,95	13,27	84,40	0,0004
20	0,2	80	60	50	30	4	20	11,18	8,66	0,32	170,82	1	34,64	8,66	272,51	1	13,72	3,43	186,59	0,3152

[a]penalty=(Teunter's approximated $TC(Q_p,Q_r)$ - proposed optimal $TC(Q_p,Q_r)$)/ Teunter's approximated $TC(Q_p,Q_r)$

Table 1. Policy (1, *R*)

Data								Teunter's exact solution				Teunter's approximated solution				Proposed optimal solution				% Cost Deviation
d	f	p	r	K_p	K_r	h_r	h_s	Q_p	Q_r	P	$TC(Q_p,Q_r)$	P	Q_p	Q_r	$TC(Q_p,Q_r)$	P	Q_p	Q_r	$TC(Q_p,Q_r)$	Penalty[a]
1000	0,8	5000	3000	20	5	2	10	70,71	34,30	0,12	346,38	1	70,71	282,8	1088,94	1	18,63	74,54	536,66	0,5072
1000	0,2	4000	2500	10	5	4	6	66,67	21,32	1,28	333,81	1	66,67	16,67	336,67	1	71,46	17,87	335,86	0,0024
500	0,3	1000	700	10	10	5	8	50,00	25,50	1,19	257,66	1	50,00	21,43	259,44	1	54,13	23,20	258,62	0,0032
500	0,3	2000	1000	20	12	10	12	47,14	18,70	0,93	489,55	1	47,14	20,20	490,13	1	45,72	19,60	489,90	0,0005
800	0,5	2400	1500	20	8	4	15	56,57	31,54	0,56	485,76	1	56,57	56,57	521,37	1	44,26	44,26	506,07	0,0293
1000	0,7	3000	2000	20	20	2	10	77,46	76,38	0,42	521,53	1	77,46	180,7	666,15	1	42,64	99,49	562,85	0,1551
20	0,8	50	35	30	20	5	6	18,26	11,58	0,16	68,41	1	18,26	73,03	191,76	1	4,49	17,98	89,01	0,5358
20	0,8	100	50	30	25	2	7	14,64	13,02	0,22	161,41	1	14,64	58,55	161,41	1	4,69	18,76	93,81	0,4188
20	0,2	80	60	50	30	4	20	11,50	6,12	2,12	177,82	2	11,55	5,77	177,82	2	11,70	5,85	177,81	0,0000

[a]penalty=(Teunter's approximated $TC(Q_p,Q_r)$ - proposed optimal $TC(Q_p,Q_r)$)/ Teunter's approximated $TC(Q_p,Q_r)$

Table 2. Policy (*P*, 1)

On Perron-Frobenius property of matrices having some negative entries

Dimitrios Noutsos[1]

Abstract

We extend the theory of nonnegative matrices to the matrices that have some negative entries. We present and prove some properties which give us information, when a matrix possesses a Perron-Frobenius eigenpair. We apply also this theory by proposing the Perron-Frobenius splitting for the solution of the linear system $Ax = b$ by classical iterative methods. Perron-Frobenius splittings constitute an extension of the well known regular splittings, weak regular splittings and nonnegative splittings. Convergence and comparison properties are given and proved.

AMS (MOS) Subject Classifications: Primary 65F10
Keywords: Perron-Frobenius theorem, nonnegative matrices, Perron-Frobenius splitting
Running Title: On Perron-Frobenius property

1 Introduction

In 1907, Perron [14] proved that the dominant eigenvalue of a matrix with positive entries is positive and the corresponding eigenvector could be chosen to be positive. With the term *dominant eigenvalue* we mean the eigenvalue which corresponds to the spectral radius. Later in 1912, Frobenius [7] extended this result to irreducible nonnegative matrices. Since then the well known *Perron-Frobenius* theory has been developed, for nonnegative matrices and the well known *Regular, Weak Regular* and *Nonnegative Splittings* have been introduced and developed for the solution of large sparse linear systems by iterative methods (Varga [16], Young [20], Berman and Plemmons [2], Bellman [1], Woźnicki [18], Csordas and Varga [5], Neumann and Plemmons [10], Miller and Neumann [9], Marek and Szyld [8], Woźnicki [19], Climent and Perea [4]). (An excellent account of all sorts of splittings can be found in Nteirmentzidis [12]). Such linear systems are yielded from the discretisation of elliptic and parabolic partial differential equations, from integral equations, from Markov chains and from other applications (see, e.g., [2]). In 1985, O'Leary and White [13] introduced the theory of Multisplittings which is very useful for the solution of linear systems on parallel computer architectures. Since then many researchers, based on their theory, have proposed various Multisplitting techniques (Neumann and Plemmons [11], Bru, Elsner and Neumann [3], Elsner [6], White [17] and others).

Recently, Tarazaga, Raydan and Hurman [15], have given a sufficient condition that guarantees the existence of the Perron-Frobenius eigenpair, for the class of symmetric matrices

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which have some negative entries. Their result was obtained by studying some convex and closed cones of matrices.

It is obvious, from the continuity of the eigenvalues and the entries of the eigenvectors, as functions of the entries of matrices, that the Perron-Frobenius theory may hold also in the case where the matrix has some absolutely small negative entries. This observation brings up some questions. E.g., How small could these entries be? What is their distribution? When such a matrix looses the Perron-Frobenius property? These questions are very difficult to answer. Tarazaga et al in [15] gave a partial answer to the first question by providing a sufficient condition for the symmetric matrix case.

In this paper the behavior of such matrices is studied. Sufficient and necessary conditions as well as monotonicity properties are stated and proved, for the general case of real matrices. So, we answer implicitly the above questions by extending the Perron-Frobenius theory of nonnegative matrices to the class of matrices that possess the Perron-Frobenius property. Finally, we apply this theory by introducing the *Perron-Frobenius splitting* for the solution of linear systems by classical iterative methods. This splitting is an extension and a generalization of the well known regular, weak regular and nonnegative splittings. We also present and prove convergence and comparison properties for the proposed splitting.

This work is organized as follows: In Section 2 the main results of the extension of the Perron-Frobenius theory are stated and proved. In Section 3 we propose the Perron-Frobenius splitting and give convergence and comparison properties based on it. As the theory is being developed, various numerical examples are given in the text to illustrate it.

2 Extension of the Perron-Frobenius theory

We begin with our theory by giving two definitions:

Definition 2.1 *A matrix $A \in I\!R^{n,n}$ possesses the Perron-Frobenius property if its dominant eigenvalue λ_1 is positive and the corresponding eigenvector $x^{(1)}$ is nonnegative.*

Definition 2.2 *A matrix $A \in I\!R^{n,n}$ possesses the strong Perron-Frobenius property if its dominant eigenvalue λ_1 is positive, simple ($\lambda_1 > |\lambda_i|$, $i = 2, 3, \cdots, n$) and the corresponding eigenvector $x^{(1)}$ is positive.*

It is noted that Definition 2.1 is the most general of the relevant ones given so far. The analogous definition in the well known Perron-Frobenius theory is that for nonnegative matrices. On the other hand, in Definition 2.2 a subset of matrices of Definition 2.1 is defined, which is analogous to that of irreducible and primitive nonnegative matrices. The next two theorems give sufficient and necessary conditions for the second class of matrices.

Theorem 2.1 *For a symmetric matrix $A \in I\!R^{n,n}$ the following properties are equivalent:*
i) A possesses the strong Perron-Frobenius property.
ii) There exists an integer $k_0 > 0$ such that $A^k > 0 \;\forall\; k \geq k_0$.

Proof: $(i \Rightarrow ii)$: Since A possesses the strong Perron-Frobenius property, its eigenvalues can be ordered as follows:

$$\lambda_1 = \rho(A) > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|,$$

where λ_1 is a simple eigenvalue with the corresponding eigenvector $x^{(1)} \in I\!R^n$ being positive. We choose an arbitrary nonnegative vector $x^{(0)} \in I\!R^n$ with $\|x^{(0)}\|_2 = 1$. We expand $x^{(0)}$ as a linear combination of the eigenvectors of A: $x^{(0)} = \sum_{i=1}^{n} c_i x^{(i)}$. Since A is symmetric the eigenvectors constitute an orthogonal basis. So, the coefficients c_i's are the inner products $c_i = (x^{(0)}, x^{(i)})$, $i = 1, 2, \cdots, n$, which means that $c_1 > 0$. We apply now the theorem of the Power method. So, the limit of $A^k x^{(0)}$ tends to the eigenvector $x^{(1)}$ as k tends to infinity. This means that for a certain $x^{(0)} \geq 0$ there exists an m such that $A^k x^{(0)} > 0$ for all $k \geq m$. If we choose the largest of all m's over all initial choices $x^{(0)} \geq 0$, specifically

$$k_0 = \max_{0 \leq x^{(0)} \in I\!R^n,\ \|x^{(0)}\|_2 = 1} \left\{ m \mid Ax^k > 0 \ \forall\ k \geq m \right\},$$

we take that for all $x^{(0)} \geq 0$, $A^k x^{(0)} > 0$ for all $k \geq k_0$, which proves our assertion.

$(ii \Rightarrow i)$: From the Perron-Frobenius theory of nonnegative matrices, the assumption $A^k > 0$ means that the dominant eigenvalue of A^k is positive and simple while the corresponding eigenvector is positive. It is well known that the matrix A has as eigenvalues the k^{th} roots of those of A^k with the same eigenvectors. Since it happens $\forall k \geq k_0$, A possesses the strong Perron-Frobenius property. □

Theorem 2.2 *For a matrix $A \in I\!R^{n,n}$ the following properties are equivalent:*
i) Both matrices A and A^T possess the strong Perron-Frobenius property.
ii) There exists an integer $k_0 > 0$ such that $A^k > 0$ for all $k \geq k_0$.

Proof: $(i \Rightarrow ii)$: Let $A = XDX^{-1}$ be the Jordan canonical form of the matrix A. We assume that the simple eigenvalue $\lambda_1 = \rho(A)$ is the first diagonal entry of D. So the Jordan canonical form can be written as

$$A = [x^{(1)} | X_{n,n-1}] \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & D_{n-1,n-1} \end{array} \right] \left[\begin{array}{c} y^{(1)^T} \\ \hline Y_{n-1,n} \end{array} \right], \tag{2.1}$$

where $y^{(1)^T}$ and $Y_{n-1,n}$ are the first row and the matrix formed by the last $n-1$ rows of X^{-1}, respectively. Since A possesses the strong Perron-Frobenius property, the eigenvector $x^{(1)}$ is positive. From (2.1), the block form of A^T is

$$A^T = [y^{(1)} | Y^T_{n-1,n}] \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & D^T_{n-1,n-1} \end{array} \right] \left[\begin{array}{c} x^{(1)^T} \\ \hline X^T_{n,n-1} \end{array} \right]. \tag{2.2}$$

The matrix $D^T_{n-1,n-1}$ is the block diagonal matrix formed by the transposes of all Jordan blocks except λ_1. It is obvious that there exists a permutation matrix $P \in I\!R^{n-1,n-1}$ such that the associated permutation transformation on the matrix $D^T_{n-1,n-1}$ transposes all the Jordan blocks. So, $D_{n-1,n-1} = P^T D^T_{n-1,n-1} P$ and relation (2.2) takes the form:

$$\begin{aligned} A^T &= [y^{(1)} | Y^T_{n-1,n}] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P \end{array} \right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P^T \end{array} \right] \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & D^T_{n-1,n-1} \end{array} \right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P \end{array} \right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P^T \end{array} \right] \left[\begin{array}{c} x^{(1)^T} \\ \hline X^T_{n,n-1} \end{array} \right] \\ &= [y^{(1)} | Y'^T_{n-1,n}] \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & D_{n-1,n-1} \end{array} \right] \left[\begin{array}{c} x^{(1)^T} \\ \hline X'^T_{n,n-1} \end{array} \right], \end{aligned} \tag{2.3}$$

where $Y'^T_{n-1,n} = Y^T_{n-1,n}P$ and $X'^T_{n,n-1} = P^T X^T_{n,n-1}$. The last relation is the Jordan canonical form of A^T which means that $y^{(1)}$ is the eigenvector corresponding to the dominant eigenvalue λ_1. Since A^T possesses the strong Perron-Frobenius property, $y^{(1)}$ is a positive vector or a negative one. Since $y^{(1)^T}$ is the first row of X^{-1} we have that $(y^{(1)}, x^{(1)}) = 1$ implying that $y^{(1)}$ is a positive vector.
We return now to the Jordan canonical form (2.1) of A and form the power A^k

$$A^k = [x^{(1)}|X_{n,n-1}] \left[\begin{array}{c|c} \lambda_1^k & 0 \\ \hline 0 & D^k_{n-1,n-1} \end{array}\right] \left[\begin{array}{c} y^{(1)^T} \\ \hline Y_{n-1,n} \end{array}\right]$$

or

$$\frac{1}{\lambda_1^k} A^k = [x^{(1)}|X_{n,n-1}] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \frac{1}{\lambda_1^k} D^k_{n-1,n-1} \end{array}\right] \left[\begin{array}{c} y^{(1)^T} \\ \hline Y_{n-1,n} \end{array}\right].$$

Since λ_1 is the simple dominant eigenvalue, we get that

$$\lim_{k\to\infty} \frac{1}{\lambda_1^k} D^k_{n-1,n-1} = 0.$$

So,

$$\lim_{k\to\infty} \frac{1}{\lambda_1^k} A^k = x^{(1)} y^{(1)^T} > 0.$$

The last relation means that there exists an integer $k_0 > 0$ such that $A^k > 0$ for all $k \geq k_0$ and the first part of Theorem is proved.

($ii \Rightarrow i$): From the Perron-Frobenius theory of nonnegative matrices, the assumption $A^k > 0$ means that the dominant eigenvalue of A^k is positive and simple while the corresponding eigenvector is positive. Considering the Jordan canonical form of A^k, $\forall k \geq k_0$, we get that the matrix A has as the dominant eigenvalue the positive k^{th} root of the one of A^k with the same eigenvector. So, A possesses the strong Perron-Frobenius property. The proof for the matrix A^T is the same by taking $(A^k)^T = (A^T)^k > 0$. □

We observe that Theorem 2.1 is a special case of Theorem 2.2. Nevertheless, it is stated and proved since the proof is quite different and easier than that of Theorem 2.2.

In the sequel some statements with necessary conditions only follow.

Theorem 2.3 *If $A^T \in \mathbb{R}^{n,n}$ possesses the Perron-Frobenius property, then either*

$$\sum_{j=1}^{n} a_{ij} = \rho(A) \quad \forall\, i = 1(1)n, \tag{2.4}$$

or

$$\min_i \left(\sum_{j=1}^{n} a_{ij}\right) \leq \rho(A) \leq \max_i \left(\sum_{j=1}^{n} a_{ij}\right). \tag{2.5}$$

Moreover, if A^T possesses the strong Perron-Frobenius property, then both inequalities in (2.5) are strict.

Proof: Let that $(\rho(A), y)$ is the Perron-Frobenius eigenpair of the matrix A^T and $\xi \in \mathbb{R}^n$ is the vector of ones $\left(\xi = (1\ 1\ \cdots\ 1)^T\right)$. We form the product $y^T A\xi$:

$$y^T A\xi = y^T \begin{pmatrix} \sum_{j=1}^n a_{1j} \\ \sum_{j=1}^n a_{2j} \\ \vdots \\ \sum_{j=1}^n a_{nj} \end{pmatrix} = \sum_{i=1}^n \left(y_i \sum_{j=1}^n a_{ij} \right) \leq \max_i \left(\sum_{j=1}^n a_{ij} \right) \sum_{i=1}^n y_i. \tag{2.6}$$

Similarly, we have that

$$y^T A\xi = \sum_{i=1}^n \left(y_i \sum_{j=1}^n a_{ij} \right) \geq \min_i \left(\sum_{j=1}^n a_{ij} \right) \sum_{i=1}^n y_i. \tag{2.7}$$

On the other hand we get

$$y^T A\xi = \xi^T A^T y = \rho(A)\xi^T y = \rho(A) \sum_{i=1}^n y_i. \tag{2.8}$$

Relations (2.6), (2.7) and (2.8) give us relation (2.5). It is obvious that the inequalities in (2.5) become equalities if $\max_i \left(\sum_{j=1}^n a_{ij}\right) = \min_i \left(\sum_{j=1}^n a_{ij}\right)$, which proves the equality (2.4). It is also obvious that the inequalities in (2.6) and (2.7) become strict if $y > 0$. So, the inequalities in (2.5) become strict if A^T possesses the strong Perron-Frobenius property. □

Note that it is necessary to have $\max_i \left(\sum_{j=1}^n a_{ij}\right) > 0$, otherwise Theorem 2.3 does not hold and so, A^T does not possess the Perron-Frobenius property. On the other hand, it is not necessary to have $\min_i \left(\sum_{j=1}^n a_{ij}\right) \geq 0$ as is shown in the following example.

Example 2.1 Let

$$A = \begin{pmatrix} 1 & 1 & -3 \\ -4 & 1 & 1 \\ 8 & 5 & 8 \end{pmatrix}.$$

The vector of the row sums of A is $(-1\ \ -2\ \ 21)^T$, while A^T possesses the strong Perron-Frobenius property with the Perron-Frobenius eigenpair: $\left(6.868\ ,\ (0.4492\ \ 0.6225\ \ 0.6408)^T\right)$.

By interchanging the roles of A and A^T, Theorem 2.3 gives an analogous result for the column sums. This is presented in the following corollary.

Corollary 2.1 *If $A \in \mathbb{R}^{n,n}$ possesses the Perron-Frobenius property, then either*

$$\sum_{i=1}^n a_{ij} = \rho(A) \quad \forall\, j = 1(1)n, \tag{2.9}$$

or

$$\min_j \left(\sum_{i=1}^n a_{ij} \right) \leq \rho(A) \leq \max_j \left(\sum_{i=1}^n a_{ij} \right). \tag{2.10}$$

Moreover, if A possesses the strong Perron-Frobenius property, then both inequalities in (2.10) are strict.

We define now the space $\mathcal{P}$ of all vectors $x \geq 0$ with at least one component being positive and its subspace $\mathcal{P}^*$, the hyperoctant of vectors $x > 0$. Then, the previous results are generalized as follows.

Theorem 2.4 *If $A^T \in \mathbb{R}^{n,n}$ possesses the Perron-Frobenius property and $x \in \mathcal{P}^*$, then either*

$$\frac{\sum_{j=1}^n a_{ij}x_j}{x_i} = \rho(A) \quad \forall\, i = 1(1)n, \tag{2.11}$$

or

$$\min_i \left(\frac{\sum_{j=1}^n a_{ij}x_j}{x_i}\right) \leq \rho(A) \leq \max_i \left(\frac{\sum_{j=1}^n a_{ij}x_j}{x_i}\right). \tag{2.12}$$

Moreover, if A^T possesses the strong Perron-Frobenius property, then both inequalities in (2.12) are strict and

$$\sup_{x\in\mathcal{P}^*}\left\{\min_i \left(\frac{\sum_{j=1}^n a_{ij}x_j}{x_i}\right)\right\} = \rho(A) = \inf_{x\in\mathcal{P}^*}\left\{\max_i \left(\frac{\sum_{j=1}^n a_{ij}x_j}{x_i}\right)\right\}. \tag{2.13}$$

Proof: Let $x \in \mathcal{P}^*$. We define the diagonal matrix $D = \text{diag}(x_1, x_2, \cdots, x_n)$ and consider the similarity transformation $B = D^{-1}AD$ (see Varga [16], Theorem 2.2). Then the entries of B are $b_{ij} = \frac{a_{ij}x_j}{x_i}$. Since B is produced from A by a similarity transformation and D and D^{-1} are both nonnegative matrices, we obtain that B^T possesses also the Perron-Frobenius property. As a consequence we have

$$\sup_{x\in\mathcal{P}^*}\left\{\min_i \left(\frac{\sum_{j=1}^n a_{ij}x_j}{x_i}\right)\right\} \leq \rho(A) \leq \inf_{x\in\mathcal{P}^*}\left\{\max_i \left(\frac{\sum_{j=1}^n a_{ij}x_j}{x_i}\right)\right\}, \tag{2.14}$$

which implies (2.12). We choose now the Perron-Frobenius eigenvector y in the place of x. It is easily seen that inequalities (2.12) become equalities, which means that those in (2.14) become also equalities and the proof is complete. □

By interchanging the roles of A and A^T, Theorem 2.4 gives us analogous results for the column sums stated in the corollary below.

Corollary 2.2 *If $A \in \mathbb{R}^{n,n}$ possesses the Perron-Frobenius property and $x \in \mathcal{P}^*$, then either*

$$\frac{\sum_{j=1}^n a_{ji}x_j}{x_i} = \rho(A) \quad \forall\, i = 1(1)n, \tag{2.15}$$

or

$$\min_i \left(\frac{\sum_{j=1}^n a_{ji}x_j}{x_i}\right) \leq \rho(A) \leq \max_i \left(\frac{\sum_{j=1}^n a_{ji}x_j}{x_i}\right). \tag{2.16}$$

Moreover, if A^T possesses the strong Perron-Frobenius property, then both inequalities in (2.12) are strict and

$$\sup_{x\in\mathcal{P}^*}\left\{\min_i \left(\frac{\sum_{j=1}^n a_{ji}x_j}{x_i}\right)\right\} = \rho(A) = \inf_{x\in\mathcal{P}^*}\left\{\max_i \left(\frac{\sum_{j=1}^n a_{ji}x_j}{x_i}\right)\right\}. \tag{2.17}$$

In the sequel we give some monotonicity properties concerning the dominant eigenvalue in the case where the matrices possess the Perron-Frobenius property. It is well known that the eigenvalues and the entries of the eigenvectors are continuous functions of the entries of a matrix A. So, if A possesses the strong Perron-Frobenius property, then a perturbation of A, $\tilde{A} = A + E$ provided $\|E\|$ is small enough, possesses also the strong Perron-Frobenius property. It is also well known, from the theory of nonnegative matrices, that the dominant eigenvalue of a nonnegative matrix A is a nondecreasing function of the entries of A, when A is reducible, while if A is an irreducible matrix, it is a strictly increasing function. Then two questions come up: What happens to the monotonicity in case the matrices possess the Perron-Frobenius property? Does the property of "possessing the Perron-Frobenius property" still hold when the entries of A increase, as it does in the nonnegative case? Unfortunately, the answer to the second question is not positive. It depends on the direction in which we increase the entries, as we will see later. First we give some properties which provide an answer to the first question.

Theorem 2.5 *If the matrices $A, B \in I\!R^{n,n}$ are such that $A \leq B$, and both A and B^T possess the Perron-Frobenius property (or both A^T and B possess the Perron-Frobenius property), then*

$$\rho(A) \leq \rho(B). \tag{2.18}$$

Moreover, if the above matrices possess the strong Perron-Frobenius property and $A \neq B$, then the inequality in (2.18) is strict.

Proof: Let $x \geq 0$ be the Perron-Frobenius eigenvector of A associated with the dominant eigenvalue λ_A and let $y \geq 0$ be the Perron-Frobenius eigenvector of B^T associated with the dominant eigenvalue λ_B. Then the following equalities hold

$$y^T A x = \lambda_A y^T x, \quad y^T B x = \lambda_B y^T x.$$

Since $A \leq B$, we can write $B = A + C$, where $C \geq 0$. So,

$$y^T B x = y^T (A + C) x = y^T A x + y^T C x \geq y^T A x.$$

Assuming that $y^T x > 0$, the above relations imply that $\lambda_B \geq \lambda_A$. The case where $y^T x = 0$ is covered by using a continuity argument. For this we consider the matrices A' and B' which are small perturbations of the matrices A and B, respectively, such that for the corresponding perturbed eigenvectors we will have $y'^T x' > 0$. The above inequality holds for the perturbed eigenvalues and because of the continuity the same property holds for the eigenvalues of A and B. It is obvious that if we follow the same reasoning we can obtain the same result in case both A^T and B possess the Perron-Frobenius property. It is also obvious that the inequality becomes strict in case the associated Perron-Frobenius properties are strong. □

We note that the above property does not guarantee the existence of the Perron-Frobenius property for an intermediate matrix C ($A \leq C \leq B$) and does not give any information about $\rho(C)$.

Theorem 2.6 *Let (i) $A^T \in \mathbb{R}^{n,n}$ possess the Perron-Frobenius property and $x \geq 0$ $(x \neq 0)$ be such that $Ax - \alpha x \geq 0$ for a constant $\alpha > 0$ or (ii) $A \in \mathbb{R}^{n,n}$ possess the Perron-Frobenius property and $x \geq 0$ $(x \neq 0)$ be such that $x^T A - \alpha x^T \geq 0$ for a constant $\alpha > 0$. Then*

$$\alpha \leq \rho(A). \tag{2.19}$$

Moreover, if $Ax - \alpha x > 0$ or $x^T A - \alpha x^T > 0$, then the inequality in (2.19) is strict.

Proof: For hypothesis (i), let $y \geq 0$ be the Perron-Frobenius eigenvector of A associated with $\rho(A)$. Then, the following equivalence holds

$$y^T(Ax - \alpha x) \geq 0 \Longleftrightarrow (\rho(A) - \alpha)y^T x \geq 0.$$

If $y^T x > 0$, then the inequality (2.19) holds. In the case where $y^T x = 0$ we recall the perturbation argument used in Theorem 2.5 to prove the validity of (2.19). If $Ax - \alpha x > 0$, the above inequalities become strict and therefore (2.19) becomes strict. For hypothesis (ii) the proof is similar. □

The above theorem is an extension of Corollary 3.2 given by Marek and Szyld in [8], for nonnegative matrices. The following theorem is also an extension of Lemma 3.3 of the same paper [8].

Theorem 2.7 *Let (i) $A^T \in \mathbb{R}^{n,n}$ possess the Perron-Frobenius property and $x > 0$ be such that $\alpha x - Ax \geq 0$ for a constant $\alpha > 0$ or (ii) $A \in \mathbb{R}^{n,n}$ possess the Perron-Frobenius property and $x > 0$ be such that $\alpha x^T - x^T A \geq 0$ for a constant $\alpha > 0$. Then*

$$\rho(A) \leq \alpha. \tag{2.20}$$

Moreover, if $\alpha x - Ax > 0$ or $\alpha x^T - x^T A > 0$, then the inequality in (2.20) becomes strict.

Proof: As in the previous theorem we give the proof only for hypothesis (i). Let $y \geq 0$ be the Perron-Frobenius eigenvector of A associated with $\rho(A)$. Then, we have

$$y^T(\alpha x - Ax) \geq 0 \Longleftrightarrow (\alpha - \rho(A))y^T x \geq 0.$$

Since $x > 0$ we have that $y^T x > 0$ and the inequality (2.20) holds. If $\alpha x - Ax > 0$, the above inequalities become strict and therefore (2.20) becomes strict. □

We remark that the condition $x > 0$ is necessaty. This is because for $x \geq 0$ such that $Ax = 0$, the condition $\alpha x - Ax \geq 0$ holds for any $\alpha \geq 0$, but the inequality (2.20) is not true for any $\alpha \geq 0$.

We give now two monotonicity properties depending on the direction in which the entries of a matrix increase.

Theorem 2.8 *Let $A \in I\!R^{n,n}$ possess the Perron-Frobenius property with $x \geq 0$ the associated eigenvector. Then, for the matrix B such that*

$$B = A + \epsilon x y^T, \quad \epsilon > 0, \ y \geq 0 \tag{2.21}$$

there holds

$$\rho(A) \leq \rho(B). \tag{2.22}$$

Moreover, if A possesses the strong Perron-Frobenius property and $y \geq 0$ ($y \neq 0$), then inequality in (2.22) becomes strict.

Proof: By post-multiplying (2.21) by x we obtain

$$Bx = (A + \epsilon x y^T)x = \left(\rho(A) + \epsilon y^T x\right) x$$

which means that $\rho(A) + \epsilon y^T x$ is an eigenvalue of B. Since $\epsilon y^T x \geq 0$ we take the inequality (2.22). The analogous proof for the strict case is obvious and is omitted. □

It is obvious that an analogous property could be given by considering that A^T possesses the Perron-Frobenius property. However, we have to remark that the above property does not guarantee the existence of the Perron-Frobenius property for the matrix B. To do this we give the following statement.

Theorem 2.9 *Let $A \in I\!R^{n,n}$ be such that both A and A^T possess the strong Perron-Frobenius property with x and y being the associated eigenvectors, respectively. Then, for the matrix B such that*

$$B = A + \epsilon x y^T, \quad \epsilon > 0, \tag{2.23}$$

there holds that both B and B^T possess the strong Perron-Frobenius property and

$$\rho(A) < \rho(B). \tag{2.24}$$

Proof: The proof of the strict inequality (2.24) is obtained from Theorem 2.8 and from the fact that $x, y > 0$. To prove the existence of the strong Perron-Frobenius property of B and B^T we use Theorem 2.2. We form $B^k = (A + \epsilon x y^T)^k$ and expand it into a sum of products of the matrices A and xy^T with the first term being A^k. Since $Axy^T = \rho(A)xy^T$ and $xy^T A = \rho(A)xy^T$, all the other $2^k - 1$ terms in the expansion, except A^k, are eventually positive scalar multiples of powers of the matrix xy^T. This means that the sum of all the other terms, except the first one, is a positive matrix. From Theorem 2.2 we have that there exists a k_0 such that $A^k > 0$ for all $k \geq k_0$. So, for this k_0 we have also $B^k > 0$ for all $k \geq k_0$, which means that both B and B^T possess the strong Perron-Frobenius property. □

We have to remark here that Theorem 2.8 gives a weak result for a dense set of directions xy^T, for all $y \geq 0$, while Theorem 2.9 gives a stronger result for precisely one direction xy^T. Based on continuity properties we can conclude that the last result is valid also for a cone of directions around xy^T.

3 Convergence theory of Perron-Frobenius splittings

In this section we define first the Perron-Frobenius splittings analogous to Regular, Weak Regular and Nonnegative splittings.

Definition 3.1 *Let $A \in I\!R^{n,n}$ be a nonsigular matrix. The splitting $A = M - N$ is*
(i) a Perron-Frobenius splitting of the first kind (kind I) if $M^{-1}N$ possesses the Perron-Frobenius property.
(ii) a Perron-Frobenius splitting of the second kind (kind II) if NM^{-1} possesses the Perron-Frobenius property.

In the sequel, for simplicity, by the term *Perron-Frobenius splitting* we mean Perron-Frobenius splitting of kind I. It is obvious from the above definition that the classes of Regular splittings, Weak Regular splittings and Nonnegative splittings belong to the class of Perron-Frobenius splittings. So, the class of Perron-Frobenius splittings is an extension of the well known, previously defined, classes. In the following, we state and prove convergence and comparison statements about this new class of splittings.

3.1 Convergence Theorems

The following theorem is an extension of the one given by Climent and Perea [4].

Theorem 3.1 *Let $A \in I\!R^{n,n}$ be a nonsigular matrix and the splitting $A = M - N$ be a Perron-Frobenius splitting, with x the Perron-Frobenius eigenvector. Then the following properties are equivalent:*
(i) $\rho(M^{-1}N) < 1$
(ii) $A^{-1}N$ possesses the Perron-Frobenius property
(iii) $\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1+\rho(A^{-1}N)}$
(iv) $A^{-1}Mx \geq x$
(v) $A^{-1}Nx \geq M^{-1}Nx$.

Proof: It can be readily found out that the matrices $A^{-1}N$ and $M^{-1}N$ are connected via the relations yielded below.

$$A^{-1}N = (M - N)^{-1}N = (I - M^{-1}N)^{-1}M^{-1}N \tag{3.25}$$

or

$$M^{-1}N = (A + N)^{-1}N = (I + A^{-1}N)^{-1}A^{-1}N. \tag{3.26}$$

The above relations imply that the matrices $A^{-1}N$ and $M^{-1}N$ have the same sets of eigenvectors with their eigenvalues being connected by

$$\mu_i = \frac{\lambda_i}{1 - \lambda_i}, \quad i = 1, 2, \cdots, n, \tag{3.27}$$

where $\lambda_i, \mu_i,\ i = 1, 2, \cdots, n$, are the eigenvalues of $M^{-1}N$ and $A^{-1}N$, respectively.

(i) $\Longrightarrow$ (ii): From $\rho(M^{-1}N) < 1$ and (3.27), there is an eigenvalue $\mu = \frac{\rho(M^{-1}N)}{1-\rho(M^{-1}N)} > 0$ of $A^{-1}N$ corresponding to the eigenvector x. Looking for a contradiction, assume that there is another eigenvalue $\mu' = \frac{\lambda'}{1-\lambda'}$ corresponding to $\rho(A^{-1}N)$. So,

$$\rho(A^{-1}N) = |\mu'| = \frac{|\lambda'|}{|1-\lambda'|} > \frac{\rho(M^{-1}N)}{1-\rho(M^{-1}N)} = |\mu|.$$

The eigenvalue λ' belongs to the disc $|z| \leq \rho(M^{-1}N)$ and $1 - \rho(M^{-1}N)$ is the distance of the point 1 from this disc. So, $|1-\lambda'| \geq 1 - \rho(M^{-1}N)$ which constitutes a contradiction.

(ii) $\Longrightarrow$ (iii): Since $A^{-1}N$ has the Perron-Frobenius eigenpair $(\rho(A^{-1}N), x)$, property (iii) follows from (3.26) by a post-multiplication by x.

(iii) $\Longrightarrow$ (i): It holds because $\rho(A^{-1}N) > 0$.

(i) $\Longleftrightarrow$ (iv): It is obvious that

$$A^{-1}Mx = (M-N)^{-1}Mx = (I - M^{-1}N)^{-1}x = \frac{1}{1-\rho(M^{-1}N)}x.$$

Since $x \geq 0,\ x \neq 0$,

$$\frac{1}{1-\rho(M^{-1}N)}x \geq x \Longleftrightarrow 0 < 1 - \rho(M^{-1}N) < 1 \Longleftrightarrow 0 < \rho(M^{-1}N) < 1.$$

(i) $\Longleftrightarrow$ (v): Considering relation (3.25) and the fact that $x \geq 0,\ x \neq 0$, we get

$$A^{-1}Nx \geq M^{-1}Nx \Longleftrightarrow \frac{\rho(M^{-1}N)}{1-\rho(M^{-1}N)}x \geq \rho(M^{-1}N)x \Longleftrightarrow \rho(M^{-1}N) < 1.$$

□

We can also state an analogous Theorem for the convergence properties of the Perron-Frobenius splittings of kind II. The proof follows the same lines as before and is omitted.

Theorem 3.2 *Let $A \in I\!R^{n,n}$ be a nonsigular matrix and the splitting $A = M - N$ be a Perron-Frobenius splitting of kind II, with x the Perron-Frobenius eigenvector. Then the following properties are equivalent:*
(i) $\rho(M^{-1}N) = \rho(NM^{-1}) < 1$
(ii) NA^{-1} *possesses the Perron-Frobenius property*
(iii) $\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1+\rho(A^{-1}N)}$
(iv) $MA^{-1}x \geq x$
(v) $NA^{-1}x \geq NM^{-1}x$.

Theorems 3.1 and 3.2 give sufficient and necessary conditions for a Perron-Frobenius splitting to be convergent. The following two theorems give only sufficient convergence conditions and constitute also extensions of the ones given by Climent and Perea [4].

Theorem 3.3 *Let $A \in I\!R^{n,n}$ be a nonsigular matrix and the splitting $A = M - N$ is a Perron-Frobenius splitting, with x the Perron-Frobenius eigenvector. If one of the following properties holds true:*
(i) There exists $y \in I\!R^n$ *such that* $A^Ty \geq 0$, $N^Ty \geq 0$ *and* $y^TAx > 0$
(ii) There exists $y \in I\!R^n$ *such that* $A^Ty \geq 0$, $M^Ty \geq 0$ *and* $y^TAx > 0$
then $\rho(M^{-1}N) < 1$.

Proof: We consider the vector z such that $y = (A^T)^{-1}z$, then the above properties are modified as follows:
(i) There exists $z \geq 0$ such that $z^T(A^{-1}N) \geq 0$, $z^Tx > 0$, and
(ii) There exists $z \geq 0$ such that $z^T(A^{-1}M) \geq 0$, $z^Tx > 0$,
respectively. We suppose that property (i) holds true. By post-multiplying by x we get

$$z^T(A^{-1}N)x = \mu z^T x \geq 0,$$

where μ is the eigenvalue of $A^{-1}N$ corresponding to the eigenvector x. So, $\mu = \frac{\rho(M^{-1}N)}{1-\rho(M^{-1}N)}$. Since $z^Tx > 0$ we get that $\mu \geq 0$, which means that $\rho(M^{-1}N) < 1$.

Let that property (ii) holds true, then by following the same steps we get

$$z^T(A^{-1}M)x = \mu' z^T x > 0$$

where $\mu' = \frac{1}{1-\rho(M^{-1}N)} > 0$ which leads to the same result. □

Moreover, we can prove that property (ii) is stronger than property (i), which means that the validity of (i) implies the validity of (ii) but the converse is not true. For this let that property (i) holds. Then

$$A^Ty \geq 0 \Longrightarrow M^Ty - N^Ty \geq 0 \Longrightarrow M^Ty \geq N^Ty \geq 0.$$

it is obvious that the converse cannot hold.

For the Perron-Frobenius splittings of kind II, the following theorem is stated.

Theorem 3.4 *Let $A \in \mathbb{R}^{n,n}$ be a nonsigular matrix and the splitting $A^T = M^T - N^T$ is a Perron-Frobenius splitting of kind II, with x the Perron-Frobenius eigenvector. If one of the following properties holds true:*
(i) There exists $y \in \mathbb{R}^n$ such that $Ay \geq 0$, $Ny \geq 0$ and $y^TA^Tx > 0$
(ii) There exists $y \in \mathbb{R}^n$ such that $Ay \geq 0$, $My \geq 0$ and $y^TA^Tx > 0$
then $\rho(M^{-1}N) < 1$.

We have to remark here that because of the sufficient conditions only, in Theorems 3.3 and 3.4, we cannot have any information about the convergence unless such a y vector exists. We show this by the following three examples.

Example 3.1

$$(i)\ A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix},\ N = \begin{pmatrix} -2 & 3 \\ -7 & 7 \end{pmatrix},\ M = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix},\ T = \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix},$$

$$A^{-1}N = \begin{pmatrix} -3 & 1 \\ -0.5 & -1 \end{pmatrix},\ A^{-1}M = \begin{pmatrix} -2 & 1 \\ -0.5 & 0 \end{pmatrix},\ \rho(T) = 4.4142,\ x = \begin{pmatrix} 0.5054 \\ 0.8629 \end{pmatrix},$$

where $T = M^{-1}N$. A vector $z \geq 0$ ($z \neq 0$) such that either $z^T(A^{-1}N) \geq 0$ or $z^T(A^{-1}M) \geq 0$ does not exist and so the splitting is **not** convergent.

$$(ii)\ A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix},\ N = \begin{pmatrix} -1 & 0 \\ 5 & -1 \end{pmatrix},\ M = \begin{pmatrix} 0 & -2 \\ 8 & -5 \end{pmatrix},\ T = \begin{pmatrix} 0.9375 & -0.125 \\ 0.5 & 0 \end{pmatrix},$$

$$A^{-1}N = \begin{pmatrix} 7 & -1 \\ 4 & -0.5 \end{pmatrix}, \; A^{-1}M = \begin{pmatrix} 8 & -1 \\ 4 & 0.5 \end{pmatrix}, \; \rho(T) = 0.8653, \; x = \begin{pmatrix} 0.8658 \\ 0.5003 \end{pmatrix}.$$

There exists no $z \geq 0$ ($z \neq 0$) such that $z^T(A^{-1}N) \geq 0$ but for $z^T = (1 \;\; 3)$ we have $z^T(A^{-1}M) \geq 0$, so the splitting **is** convergent.

$$(iii)\; A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}, \; N = \begin{pmatrix} -1 & 0 \\ 5 & -3 \end{pmatrix}, \; M = \begin{pmatrix} 0 & -2 \\ 8 & -7 \end{pmatrix}, \; T = \begin{pmatrix} 1.0625 & -0.3750 \\ 0.5 & 0 \end{pmatrix},$$

$$A^{-1}N = \begin{pmatrix} 7 & -3 \\ 4 & -1.5 \end{pmatrix}, \; A^{-1}M = \begin{pmatrix} 8 & -3 \\ 4 & -0.5 \end{pmatrix}, \; \rho(T) = 0.8390, \; x = \begin{pmatrix} 0.8590 \\ 0.5119 \end{pmatrix}.$$

There exists no $z \geq 0$ ($z \neq 0$) such that either $z^T(A^{-1}N) \geq 0$ or $z^T(A^{-1}M) \geq 0$ but the splitting **is** convergent.

We have also to remark that the strict condition $y^T Ax > 0$ is necessary. This is shown in the following example.

Example 3.2

$$A = \begin{pmatrix} 1 & -2 & -1 \\ 3 & -4 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \; N = \begin{pmatrix} -2 & 3 & 1 \\ -7 & 7 & 1 \\ 2.5 & -2 & 1 \end{pmatrix}, \; M = \begin{pmatrix} -1 & 1 & 0 \\ -4 & 3 & 2 \\ 1.5 & -1 & 2 \end{pmatrix}, \; T = \begin{pmatrix} 1 & 2 & \frac{8}{3} \\ -1 & 5 & \frac{11}{3} \\ 0 & 0 & \frac{1}{3} \end{pmatrix},$$

$$A^{-1}N = \begin{pmatrix} -3 & 1 & -2.5 \\ -0.5 & -1 & -2 \\ 0 & 0 & 0.5 \end{pmatrix}, \; \rho(T) = 4.4142, \; x = \begin{pmatrix} 0.5054 \\ 0.8629 \\ 0 \end{pmatrix}.$$

For the vector $z^T = (0 \;\; 0 \;\; 1)$ all but one of the conditions of Theorem 3.3 (i) hold. However, since $z^T x = 0$ the splitting is **not** convergent.

From Theorems 3.3 and 3.4 the corollaries below follow.

Corollary 3.1 *Let $A \in \mathbb{R}^{n,n}$ be a nonsingular matrix and the splitting $A = M - N$ be a Perron-Frobenius splitting, with x the Perron-Frobenius eigenvector. If one of the matrices $(A^{-1}N)^T$ or $(A^{-1}M)^T$ possesses also the Perron-Frobenius property with y the associated Perron-Frobenius eigenvector, such that $y^T x > 0$, then $\rho(M^{-1}N) < 1$.*

Proof: Since $y \geq 0$ and $y^T(A^{-1}N) \geq 0$ or $y^T(A^{-1}M) \geq 0$, respectively, the vector y plays the role of z in the proof of Theorem 3.3, so the splitting is convergent. □

Corollary 3.2 *Let $A \in \mathbb{R}^{n,n}$ be a nonsingular matrix and the splitting $A^T = M^T - N^T$ be a Perron-Frobenius splitting, with x the Perron-Frobenius eigenvector. If one of the matrices NA^{-1} or MA^{-1} possesses also the Perron-Frobenius property with y the associated Perron-Frobenius eigenvector, such that $y^T x > 0$, then $\rho(M^{-1}N) < 1$.*

3.2 Comparison Theorems

The following theorem is an extension of the one given by Marek and Szyld [8] for nonnegative splittings.

Theorem 3.5 *Let $A \in \mathbb{R}^{n,n}$ be a nonsingular matrix such that $A^{-1} \geq 0$. If one of the following properties holds true:*
(i) $A = M_1 - N_1$ and $A^T = M_2^T - N_2^T$ are two convergent Perron-Frobenius splittings of kind I and of kind II, respectively, with $T_1 := M_1^{-1}N_1$, $T_2^T := (M_2^{-1}N_2)^T$ and $x \geq 0$, $y \geq 0$ the associated Perron-Frobenius eigenvectors, respectively, and

$$N_2 x \geq N_1 x, \tag{3.28}$$

(ii) $A^T = M_1^T - N_1^T$ and $A = M_2 - N_2$ are two convergent Perron-Frobenius splittings of kind II and of kind I, respectively, with $T_1^T := (M_1^{-1}N_1)^T$, $T_2 := M_2^{-1}N_2$ and $y' \geq 0$, $z > 0$ the associated Perron-Frobenius eigenvectors, respectively, and

$$N_2 z \geq N_1 z, \tag{3.29}$$

then

$$\rho(T_1) \leq \rho(T_2). \tag{3.30}$$

Moreover, if $A^{-1} > 0$ and $N_2 x \neq N_1 x$, $N_2 z \neq N_1 z$, respectively, then

$$\rho(T_1) < \rho(T_2). \tag{3.31}$$

Proof: Let that property (i) holds. Then

$$A^{-1}N_2 x \geq A^{-1}N_1 x.$$

Since the above splittings are convergent, from Theorem 3.1 property (ii), we get that the matrix $A^{-1}N_1$ possesses the Perron-Frobenius property and from Theorem 3.2 property (ii), we get that the matrix $(A^{-1}N_2)^T$ possesses the Perron-Frobenius property, with x and y the Perron-Frobenius eigenvectors, respectively. So,

$$A^{-1}N_2 x - \rho(A^{-1}N_1)x \geq 0$$

and by Theorem 2.6 we get that $\rho(A^{-1}N_2) \geq \rho(A^{-1}N_1)$. Since $\rho(A^{-1}N_1) = \frac{\rho(T_1)}{1-\rho(T_1)}$, $\rho((A^{-1}N_2)^T) = \rho(A^{-1}N_2) = \frac{\rho(T_2)}{1-\rho(T_2)}$ and the fact that the function $\frac{\rho}{1-\rho}$ is an increasing function of $\rho \in (0,1)$, the result (3.30) follows. The strict inequality (3.31) becomes obvious from the fact that $A^{-1} > 0$ and $N_2 x \neq N_1 x$, $N_2 z \neq N_1 z$, respectively. The proof in case property (ii) holds is analogous, where use of Theorem 2.7 is made this time. □

We show the validity of this theorem by the following example.

Example 3.3 We consider the splittings $A = M_1 - N_1 = M_2 - N_2 = M_3 - N_3$ where

$$A = \begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -2 & 0 \\ 0 & -2 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 4 & -1.1 & 0.2 & 0 \\ -1.1 & 4 & -1 & 0 \\ 0.2 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}.$$

The splitting $A = M_1 - N_1$ is a Perron-Frobenius splitting with the Perron-Frobenius eigenpair being $(\rho(T_1), x_1) = \big(0.5345, (0.5680 \;\; 0.4212 \;\; 0.4212 \;\; 0.5680)^T\big)$. The splitting $A^T = M_2^T - N_2^T$ is a Perron-Frobenius splitting of kind II with the Perron-Frobenius eigenpair being $(\rho(T_2), y_2) = \big(0.6126, (0.6388 \;\; 0.2855 \;\; 0.3871 \;\; 0.6005)^T\big)$. Although $N_2 - N_1$ is not a nonnegative matrix, we have $(N_2 - N_1)x_1 = (0.0421 \;\; 0.3644 \;\; 0.5348 \;\; 0)^T \geq 0$. Moreover, $A^{-1} > 0$ and $N_2 x_1 \neq N_1 x_1$. So, property (i) of Theorem 3.5 holds and the inequality $\rho(T_1) < \rho(T_2)$ is confirmed. We can check that for the first two splittings, property (ii) of Theorem 3.5 also holds.
To compare the last two splittings we observe that the splitting $A = M_2 - N_2$ is a Perron-Frobenius splitting while $A = M_3 - N_3$ is a regular splitting, but properties (i) and (ii) of Theorem 3.5 do not hold. So, Theorem 3.5 does not give any information.
We have to observe here that both properties (i) and (ii) of Theorem 3.5 hold for the comparison of the first splitting with the last one, since $N_3 - N_1 \geq 0$. So, $\rho(T_1) = 0.5345 < \rho(T_3) = 0.6667$ is confirmed.

The above theorem can be extended further by replacing condition $A^{-1} \geq 0$ by a weaker one. So, we can have the following statement.

Theorem 3.6 *Let $A \in \mathbb{R}^{n,n}$ be a nonsingular matrix. If one of the following properties holds true:*
(i) $A = M_1 - N_1$ and $A^T = M_2^T - N_2^T$ are two convergent Perron-Frobenius splittings of kind I and of kind II, respectively, with $T_1 := M_1^{-1}N_1$, $T_2^T := (M_2^{-1}N_2)^T$ and $x \geq 0$, $y \geq 0$ the associated Perron-Frobenius eigenvectors, respectively, such that

$$y^T A^{-1} \geq 0, \quad y^T x > 0 \quad \text{and} \quad N_2 x \geq N_1 x, \tag{3.32}$$

(ii) $A^T = M_1^T - N_1^T$ and $A = M_2 - N_2$ are two convergent Perron-Frobenius splittings of kind II and of kind I, respectively, with $T_1^T := (M_1^{-1}N_1)^T$, $T_2 := M_2^{-1}N_2$ and $y' \geq 0$, $z > 0$ the associated Perron-Frobenius eigenvectors, respectively, such that

$$y'^T A^{-1} \geq 0, \quad y'^T z > 0 \quad \text{and} \quad N_2 z \geq N_1 z, \tag{3.33}$$

then

$$\rho(T_1) \leq \rho(T_2). \tag{3.34}$$

Moreover, if $y^T A^{-1} > 0$ and $N_2 x \neq N_1 x$ for property (i) or $y'^T A^{-1} > 0$ and $N_2 z \neq N_1 z$ for property (ii), then

$$\rho(T_1) < \rho(T_2). \tag{3.35}$$

Proof: Let that property (i) holds. Then from the first and the last inequalities of (3.32) we get

$$y^T A^{-1} N_2 x \geq y^T A^{-1} N_1 x.$$

As in Theorem 3.5, it can be implied in a similar way that both matrices $A^{-1}N_1$ and $(A^{-1}N_2)^T$ possess the Perron-Frobenius property, with x and y the Perron-Frobenius eigenvectors, respectively. So,

$$\rho(A^{-1}N_2)y^Tx - \rho(A^{-1}N_1)y^Tx \geq 0$$

and therefore $\rho(T_1) \leq \rho(T_2)$. The strict inequality (3.35) is obvious. The proof in case property (ii) holds is similar. □

Theorem 3.7 *Let $A \in I\!R^{n,n}$ be a nonsingular matrix. If one of the following properties holds true:*
(i) $A = M_1 - N_1$ and $A^T = M_2^T - N_2^T$ be two Perron-Frobenius convergent splittings of kind I and of kind II, respectively, with $T_1 := M_1^{-1}N_1$, $T_2^T := (M_2^{-1}N_2)^T$ and $x \geq 0$, $y \geq 0$ the associated Perron-Frobenius eigenvectors, respectively,

$$N_1x \geq 0 \text{ and } M_1^{-1} \geq M_2^{-1}, \tag{3.36}$$

(ii) $A^T = M_1^T - N_1^T$ and $A = M_2 - N_2$ be two Perron-Frobenius convergent splittings of kind II and of kind I, respectively, with $T_1^T := (M_1^{-1}N_1)^T$, $T_2 := M_2^{-1}N_2$ and $y' \geq 0$, $z > 0$ the associated Perron-Frobenius eigenvectors, respectively,

$$N_2z \geq 0 \text{ and } M_1^{-1} \geq M_2^{-1}, \tag{3.37}$$

then

$$\rho(T_1) \leq \rho(T_2). \tag{3.38}$$

Moreover, if $M_1^{-1} > M_2^{-1}$ and $N_1x \neq 0$, $N_2z \neq 0$, respectively, then

$$\rho(T_1) < \rho(T_2). \tag{3.39}$$

Proof: We assume that property (i) holds. Then

$$M_1x = \frac{1}{\rho(T_1)}N_1x \geq 0$$

and

$$Ax = M_1(I - T_1)x = \frac{1-\rho(T_1)}{\rho(T_1)}N_1x \geq 0.$$

By premultiplying by $M_1^{-1} - M_2^{-1} \geq 0$ we get

$$(M_1^{-1} - M_2^{-1})Ax = (I - T_1)x - (I - T_2)x = T_2x - \rho(T_1)x \geq 0.$$

By Theorem 2.6 we obtain the result (3.38). The strict inequality (3.39) is obvious and that the proof in case property (ii) holds is quite analogous. □

We observe that Theorem 3.7 provides an answer to Example 3.3 where Theorem 3.5 failed. Especially, we have $M_2^{-1} - M_3^{-1} > 0$ and $N_2x_2 \geq 0$, $N_2x_2 \neq 0$. So the strict inequality $\rho(T_2) = 0.6126 < \rho(T_3) = 0.6667$ is confirmed. It is easily checked that property (ii) of

Theorem 3.7 also holds. We also observe that both properties (i) and (ii) of Theorem 3.7 hold for the comparison of the first with the second splitting as well as the first with the last one.

As we provided an extension from Theorem 3.5 to Theorem 3.6 we can extend also Theorem 3.7 by simply replacing the condition $M_1^{-1} \geq M_2^{-1}$ by a weaker one. This is stated in the following theorem, where the proof is similar to the previous one.

Theorem 3.8 *Let $A \in I\!R^{n,n}$ be a nonsingular matrix. If one of the following holds:*
(i) $A = M_1 - N_1$ and $A^T = M_2^T - N_2^T$ are two convergent Perron-Frobenius splittings of kind I and of kind II, respectively, with $T_1 := M_1^{-1}N_1$, $T_2^T := (M_2^{-1}N_2)^T$ and $x \geq 0$, $y \geq 0$ the associated Perron-Frobenius eigenvectors, respectively, $N_1 x \geq 0$ and $y^T M_1^{-1} \geq y^T M_2^{-1}$, $y^T x > 0$,
(ii) $A^T = M_1^T - N_1^T$ and $A = M_2 - N_2$ are two convergent Perron-Frobenius splittings of kind II and of kind I, respectively, with $T_1^T := (M_1^{-1}N_1)^T$, $T_2 := M_2^{-1}N_2$ and $y' \geq 0$, $z > 0$ the associated Perron-Frobenius eigenvectors, respectively, $N_2 z \geq 0$ and $y'^T M_1^{-1} \geq y'^T M_2^{-1}$, $y'^T x > 0$, then

$$\rho(T_1) \leq \rho(T_2). \tag{3.40}$$

Moreover, if $y^T M_1^{-1} > y^T M_2^{-1}$ and $N_1 x \neq 0$ or $y'^T M_1^{-1} > y'^T M_2^{-1}$ and $N_2 z \neq 0$, respectively, then the inequality (3.40) is strict, while if $y^T M_1^{-1} = y^T M_2^{-1}$ or $y'^T M_1^{-1} = y'^T M_2^{-1}$, respectively, then the inequality (3.40) becomes an equality.

In the following example it is shown how the three previous theorems work.

Example 3.4 We consider the splittings $A = M_1 - N_1 = M_2 - N_2 = M_3 - N_3 = M_4 - N_4 = M_5 - N_5$ where

$$A = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 3 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 3 & -1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}.$$

It is easily checked that all the above splittings are convergent ones with

$$\rho(T_2) = 0 < \rho(T_1) = \rho(T_3) = \rho(T_4) = \frac{1}{3} < \rho(T_5) = 0.4472.$$

The first four splittings are Perron-Frobenius splittings while the last one is a nonnegative splitting. The splittings $A^T = M_1^T - N_1^T = M_3^T - N_3^T = M_4^T - N_4^T$ are also Perron-Frobenius splittings while the splitting $A^T = M_5^T - N_5^T$ is a nonnegative splitting. The associated Perron-Frobenius eigenvectors are:

$$x_1 = x_2 = \begin{pmatrix} 0.7071 \\ 0.7071 \\ 0 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0.8018 \\ 0.5345 \\ 0.2773 \end{pmatrix}, \quad x_4 = \begin{pmatrix} 0.4082 \\ 0.8165 \\ 0.4082 \end{pmatrix}, \quad x_5 = \begin{pmatrix} 0.6325 \\ 0.7071 \\ 0.3162 \end{pmatrix},$$

$$y_1 = y_3 = y_4 = \begin{pmatrix} 0 \\ 0.7071 \\ 0.7071 \end{pmatrix}, \quad y_5 = \begin{pmatrix} 05130 \\ 0.6882 \\ 0.5130 \end{pmatrix},$$

where by x_i and y_i we have denoted the associated Perron-Frobenius eigenvectors of kind I and of kind II, respectively. It is easily checked that A^{-1} is not a nonnegative matrix so, Theorem 3.5 cannot be applied and therefore we will try to confirm our results by applying Theorems 3.6, 3.7 or 3.8. We use the symbol $i \leftrightarrow j$ to denote the comparison of the i^{th} splitting with the j^{th} one:

$1 \leftrightarrow 2$: It is easily checked that assumptions (i) of Theorems 3.6, 3.7 and 3.8 hold, where the roles of T_1 and T_2 have been interchanged, to obtain $\rho(T_2) \leq \rho(T_1)$. Note that the strict inequality cannot be obtained from any of the above theorems.

$1 \leftrightarrow 3$: Theorems 3.6 and 3.7 cannot by applied while both assumptions (i) and (ii) of Theorem 3.8 hold with the corresponding inequalities $y_3^T M_1^{-1} \geq y_3^T M_3^{-1}$ and $y_1^T M_1^{-1} \geq y_1^T M_3^{-1}$ being equalities. So, we obtain $\rho(T_1) = \rho(T_3)$.

$3 \leftrightarrow 2$: The same properties, as in the case $1 \leftrightarrow 2$, hold. Therefore, $\rho(T_2) \leq \rho(T_3)$.

$3 \leftrightarrow 4$: The same properties, as in the case $1 \leftrightarrow 3$, hold. So, $\rho(T_3) = \rho(T_4)$.

$4 \leftrightarrow 2$: The same properties, as in the case $1 \leftrightarrow 2$, hold. Consequently, $\rho(T_2) \leq \rho(T_4)$.

$4 \leftrightarrow 5$: Both properties of Theorems 3.6, 3.7 and 3.8 are applied to give the inequality $\rho(T_4) \leq \rho(T_5)$. Moreover, we have that $y_5^T A^{-1} > 0$ and $y_5^T M_4^{-1} > y_5^T M_5^{-1}$, which gives by Theorems 3.6 and 3.8, respectively, the strict inequality $\rho(T_4) < \rho(T_5)$.

$5 \leftrightarrow 2$: From property (i) of Theorem 3.6 and the fact that $y_5^T A^{-1} > 0$ we obtain the strict inequality $\rho(T_2) < \rho(T_5)$.

We conclude this work by pointing out that the most general extensions and generalizations of the Perron-Frobenius theory for nonnegative matrices, have been introduced, stated and proved. Our theory can be applied for the solution of linear systems derived from the discretisation of elliptic and parabolic partial differential equations, from integral equations, from Markov chains and from other applications. The introduced Perron-Frobenius splittings can also be used in connection with the multisplitting techniques in order to solve linear systems of the aforementioned applications on computers of parallel architecture.

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More on Shortest and Equal Tails Confidence Intervals

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Abstract

An interesting topic in mathematical statistics is that of the construction of the confidence intervals. Two kinds of intervals which are both based on the method of the pivotal quantity are a) the Shortest Confidence Interval (SCI) and b) the Equal Tails Confidence Intervals (ETCI). The aim of this paper is i) to clarify and comment on the finding of such intervals, ii) to investigate the relation between the two kinds of intervals, iii) to point out that the existence of confidence intervals with the shortest length do not always exist, even when the distribution of the pivotal quantity is symmetric and finally iv) to give similar results when the Bayes approached is used. We believe that all these will contribute to in classroom presentation of the topic to the graduate and postgraduate students.

Key Words: Pivotal quantity; Monotonicity; Bayes; Unimodal.

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1. INTRODUCTION

Let X be a real value random variable (r.v) from the density f(x;θ) and consider the parameter θ as a fixed unknown quantity. If we seek an interval for θ, then it is well known that the standard method for obtaining confidence intervals for θ is the pivotal quantity method. (cf. Huzurbazar (1955), Guenther (1969, 1987), Dahiya and Guttman (1982), Ferentinos (1987, 1988, 1990), Juola (1993), Ferentinos and Kourouklis (1990), Kirmani (1990), Casella and Berger (2002), Rohatgi and Saleh (2001) e.t.c).

Let $Q(X_1, X_2, \ldots, X_n;\theta)$ be a pivotal quantity where $X_1, X_2, \ldots, X_n$ is a random (r.s) from the distribution of f(x;θ). The probability statement

$$P(q_1 < Q < q_2) = 1-\alpha \tag{1.1}$$

is converted (when possible) to

$$P(q_1^* < \theta < q_2^*) = 1-\alpha\,. \tag{1.2}$$

If constants q_1, q_2 in (1.1)can be found so that $(q_2^* - q_1^*)$ is minimum, then the interval $(q_1^*,\, q_2^*)$ is said to be the shortest confidence interval based on Q. Frequently $(q_2^* - q_1^*)$ can be expressed as

$$l = q_2^* - q_1^* = w(x)\varphi(q_1, q_2), \tag{1.3}$$

where the function w does not involve q_1, q_2 and φ is independent of x. In such situations minimizing $q_2^* - q_1^*$ is the same as minimizing $E(q_2^* - q_1^*)$. On the other hand if constants q_1, q_2 can be determined so that

$$P(Q < q_1) = \alpha/2 \;\text{ and }\; P(Q > q_2) = \alpha/2 \tag{1.4}$$

then the interval $(q_1^*,\, q_2^*)$ is said to be an equal tails confidence interval.
In both situations we have the same confidence interval, symbolically $CI(q_1^*, q_2^*)$, which is based on the same pivotal quantity Q. What it is different is the determination of the q_1 and q_2 (cf. previous references).

The aim of this work is to clarify and comment on problems that emerge at the process of finding, to investigate the relation of equality of length of these, to point out the non existence of SCI even when the distribution of the pivotal quantity is symmetric and finally to give similar results based on the Bayes approach.

2. MAIN RESULTS

2.1 The case where the SCI coincide with the ETCI

As it was pointed out earlier the SCI and the ETCI differ only as for the determination of q_1, q_2. An interesting question that springs up is when this determination is identical, i.e when those intervals have the same length. And reversely, if the two intervals have the same length does it characterize the distribution of the pivotal quantity? An answer to the last question is given, partially, by the work of Kirmani (1990). In this work it is shown when the ETCI minimize the length based on the pivotal quantity Q. More specifically it shown that "*... an ETCI obtained from a symmetrically distributed pivotal quantity does not necessary have minimum length unless the distribution function of the pivotal quantity is concave on the right of the point of symmetry*". Also (partly) answer in this question gives the theorem 9.3.2 in combination with exercise 9.39 in the book of Casella and Berger

(2002). More concretely if the distribution f(q) of the pivotal quantity Q is unimodal, then the interval $[q_1, q_2]$ that satisfies the relationships

$$\text{(i)} \int_{q_1}^{q_2} f(q)dq = 1-\alpha, \text{ (ii) } f(q_1) = f(q_2) > 0 \text{ and (iii) } q_1 \le q^* \le q_2, \tag{2.1}$$

where q* is the median of f(q), is the shortest among all intervals that satisfy (i). Exercise 9.39 says that if f(q) is symmetric and unimodal then for confidence intervals of the form $[q_1, q_2]$ the requirements of theorem 9.3.2 are satisfied and also q_1, q_2 are such that this to be also an ETCI.

The two approaches can be shown to be equivalent if as a point of symmetry we will take the zero point. The precedents give the spark for an overall confrontation of this subject (generalisation and fulfilment) and if the reverse is also true. So we come up with the following proposition.

Proposition 2.1 Let Q=Q(x; θ) be a pivotal quantity with p.d.f. f(q). Let also l_{SCI} and l_{ETCI} be the lengths of a SCI and ETCI, respectively, for θ based on Q. Then, if f(q) is symmetric and unimodal, $l_{SCI}=L_{ETCI}$, provided that the length l is of the form $l=c(q_2-q_1)$, c>0.

Proof:

We define the sequences of points $q_{1,k}$ and $q_{2,k}$ such that

$$\int_{-\infty}^{q_{1,k}} f(q)dq = \alpha/k \text{ and } \int_{q_{2,k}}^{+\infty} f(q)dq = (k-1)\alpha/k,\ k>1. \tag{2.2}$$

Obviously $P(q_{1,k} < Q < q_{2,k}) = 1-\alpha$. Also $F_Q(q_{1,k})=\alpha/k$ and $F_Q(q_{2,k})=1-(k-1)\alpha/k$. Hence

$$q_{1,k} = F_Q^{-1}(\alpha/k) \text{ and } q_{2,k} = F_Q^{-1}(1-(k-1)\alpha/k). \tag{2.3}$$

Since f(q) is symmetric and unimodal, minimizing the length l of the interval $[q_{1,k}, q_{2,k}]$, we get $f(q_{1,k})=f(q_{2,k})$. From this relationship we can determine the values of $q_{1,k}$ and $q_{2,k}$. Without loss of generality we can assume that $-q_{1,k} = q_{2,k}$. Now using (2.3) we get that $-F_Q^{-1}(\alpha/k) = F_Q^{-1}(1-(k-1)\alpha/k)$ which implies that

$$F_Q\left(F_Q^{-1}(\alpha/k)\right) = 1-(k-1)\alpha/k, \tag{2.4}$$

or because of the symmetry of f(q) (see Kirmani 1990)

$$F_Q\left[F_Q^{-1}(\alpha/k)\right] = 1-F_Q\left[F_Q^{-1}(\alpha/k)\right] = 1-\alpha/k. \tag{2.5}$$

From the last two relations we get that $1-\alpha/k=1-(k-1)\alpha/k$ and hence k=2. This completes the proof of the proposition 2.1.

Remarks: (i) We can get a proof of the previous proposition if we combine theorem 9.3.2 and exercise 9.39 of Casella and Berger (2002) or from the theorem of Kirmani (1990). However we believe that the previous proof she is different, sort and at straight line proof.

(ii) Proposition 2.1 has been proved for confidence intervals whose length is of the form $l=c(q_2-q_1)$. However the problem remains unsolved for confidence intervals whose length is of the form $l=c(1/q_1-1/q_2)$. It is the author's guess that if the distribution f(q) is symmetric and unimodal then the only form of the length of the confidence interval is the first one.

(iii) Interest presents the reverse of the proposition 2.1 (because it can constitutes a characterization of f(q)). That is, if $l_{SCI}=l_{ETCI}$ then f(q) is symmetric and unimodal. It is guessed that this may be the case for distributions like normal and t. Although a rigorous proof of it, it is not known (so it remains an opened problem), the following argument supports this idea. "*In the case of the SCI the q_1 and q_2 are one a function of* *the other, i.e $q_1=q_1(q_2)$ (see relation (1.3)). This is because the length l must be the shortest one. On the other hand this is not the case for a ETCI. In this case each of the q_1 and q_2 is determined independently of each other (see relations (1.4)). When we say that the two kinds of intervals coincide (i.e $l_{SCI}=l_{ETCI}$) we mean that they are determined by the same q's. That is if q_1^* and q_2^*, ($q_1^* < q_2^*$), are the points which determine the ETCI then $q_1^*=q_1$ and $q_2^*=q_2$ and hence $q_1^*=q_1^*(q_2^*)$. This implies that in the case of ETCI the q's are function of each other and their relation is linear, since the length of the interval is of the form $l=c(q_2-q_1)$. In order this to happen the distribution f(q) must be symmetric and unimodal, i.e. $q_2=q_1+c$".*

(iv) It is known that the SCI based on the pivotal quantity Q it is shortest for the specific pivotal. This means that we can find another pivotal quantity Q* which will give even a shortest interval than that based on Q. (cf. Ferentinos 1988). The question which naturally arises is how to find the pivotal quantity that gives the overall SCI. The literature does not give a clear answer on this point. Intuitively, a reasonable choice is the pivotal quantity to be a function of a sufficient statistic (only), (Guenther, 1969). Moreover, using theorem 9.3.2 of Casella and Berger (2002) we get that the CI $[q_1, q_2]$ is the shortest among all intervals that satisfy (1.1). Now from exercise 9.39 of the same authors, if f(q) is symmetric and unimodal then the previous relation it is satisfied. Thus we can state the following proposition:

Proposition 2.2 The SCI based on pivotal quantities with p.d.f. symmetric and unimodal is the overall shortest confidence interval.

2.2 Monotonicity of f(q) and l(q)

To find a SCI one can use the Lemma 2.1 in Ferentinos and Kourouklis (1990) or equivalently the theorem in Juola (1993). Usually, in most of the cases, one follows the classical minimization process under constraints. This means that one wants to minimize relation (1.3) subject to condition (1.2). The most frequently cases are those where the function $\varphi(q_1, q_2)$ is of the form (q_2-q_1) or $(1/q_1-1/q_2)$. In those cases the minimization problem leads, respectively, to the following relations

$$\text{(i) } f(q_1)=f(q_2) \text{ and (ii) } q_1^2 f(q_1)=q_2^2 f(q_2), \tag{2.6}$$

or we decide based on the monotonicity of the f(q).

If f(q) is symmetric and unimodal then (w. l. o g. we can assume that $-q_1=q_2$) the quantities q_1 and q_2 are determined from the relation (2.6) (i). However if f(q) is monotonic then it is almost impossible to use relations (2.6). In those cases the minimization problem it is based on the monotonicity of the length l. From this process results the following interest proposition (characterization) for the length l, which depends on the monotonicity of f(q) and facilitates the determination of q_1 and q_2 guiding us to the right direction with respect to the differentiation of q_1 or q_2 (see comment on example 2.2).

Proposition 2.3 Let $Q=Q(x;\theta)$ be a pivotal quantity for a parameter θ with p.d.f. f(q). For the 100(1-α)%CI for θ based on Q of the form $P(q_1<Q<q_2)=1-\alpha$ with length $l_1=w_1(x)(q_2-q_1)$ or $l_2=w_2(x)(1/q_1-1/q_2)$ we have:

(i) if f(q) is a strictly increasing p.d.f. on $[k_1, k_2]$, $k_i \in R$, (i=1, 2) then $l_i(q)$ is strictly decreasing on $[k_1, k_2]$.

(ii)) if f(q) is a strictly decreasing p.d.f. on $[k_1, k_2]$, $k_i \in R$, (i=1, 2) then $l_1(q)$ is strictly increasing on $[k_1, k_2]$.

Proof.

(i) It is easy to see that the minimum of $l_1(q)$ subject to (1.1) occurs for those values of q_1 and q_2 which satisfy the relation

$$\frac{dl_1}{dq_2} = w_1(x)\left(1 - \frac{f(q_2)}{f(q_1)}\right). \tag{2.7}$$

The fact that f(q) is strictly increasing implies that $f(q_1) \neq f(q_2)$. More ever if $q_1 < q_2$ then $f(q_1) < f(q_2)$. Thus from (2.7) and given that $w_1(x) > 0$ we get that $dl_1/dq_2 < 0$. This means that $l_1(q)$ is strictly increasing on some interval $[k_1, k_2]$. Hence the q_1 and q_2, for a SCI, are determined by the relations

$$q_2 = k_2 \text{ and } \int_{q_1}^{k_1} f(q)dq = 1 - \alpha. \tag{2.8}$$

For the case $l_2 = w_2(x)(1/q_1 - 1/q_2)$ we have

$$\frac{dl_2}{dq_2} = w_2(x)\frac{q_1^2 f(q_1) - q_2^2 f(q_2)}{q_1^2 q_2^2 f(q_1)}.$$

Now since $w_2(x) > 0$, $q_1 < q_2$ and $f(q_1) < f(q_2)$ we obtain that $dl_2/dq_2 < 0$. Thus l_2 is strictly decreasing on some interval $[k_1, k_2]$. The q_1 and q_2 can be determined from the relations (2.8).

(ii) Working in a way similar to that in (i) we can show that in the case of l_1 the quantities q_1 and q_2 are determined from the relations

$$q_1 = k_1 \text{ and } \int_{k_1}^{q_2} f(q)dq = 1 - \alpha.$$

In the case of l_2 we can not say anything about the sign of dl_2/dq_1. The quantities q_1 and q_2 are determined from the relation $q_1^2 f(q_1) = q_2^2 f(q_2)$.

Remark: From (i) of the previous proposition we have that when f(q) is strictly increasing then both l_1 and l_2 are strictly decreasing. This means that the SCI (if it exists) take place on the upper point of the interval where Q is defined, that is the point k_2. Hence the derivation of l(q) should be with respect the q_2. In the opposite case the derivation should be with respect the q_1. Another way for expressing the same thing is to set $q_1 = q$ and $q_2 = \delta(q)$ $(q < \delta(q))$.

We will clarify the previous proposition with the following examples.

Example 2.1 (Ferentinos 1990) Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with density $f(x,\theta) = g(x)/h(\theta)$, $a(\theta) \leq x \leq b(\theta)$. If $\hat{\theta}$ is a sufficient statistic for θ, then it is known that the quantity $Q = h(\hat{\theta})/h(\theta)$ is a pivotal quantity with distribution (Huzurbazar 1955) $f(q) = nq^{n-1}$, $0 \leq q \leq 1$. Obviously f(q) is strictly increasing on [0, 1] for n>1. The CI based on Q can be found from the relation $P(q_1 < Q, q_2) = 1 - \alpha$, from which we get that $l = h(\hat{\theta})(1/q_1 - 1/q_2)$. So, from proposition 2.2, the length of the interval l is strictly decreasing and hence the SCI is obtained on the points $q_2 = 1$ and q_1 given from the relation $\int_{q_1}^{1} f(q)dq = 1 - \alpha$. Finally the SCI for $h(\theta)$ we get is the well known one $h(\hat{\theta}) \leq h(\theta) \leq h(\hat{\theta})\alpha^{-1/n}$.

Example 2.2 (Guenther 1969) Let $X_1, X_2, ..., X_n$ be a random sample from the distribution $f(x, \theta)=e^{-(x-\theta)}$, $x\geq\theta$. If $T=\min X_i$ (i=1, 2, …, n) is the sufficient statistic for the parameter θ then $Q=2n(T-\theta)$ is a pivotal quantity with p.d.f $f(q)=(1/2)e^{-q/2}$, $q\geq0$. It is clear that f(q) is strictly decreasing on [0, ∞] and hence, according to proposition 2.2 (ii), l(q) is strictly increasing on [0, ∞]. Thus the SCI will be given from the points q_1 and q_2, where $q_1=0$ and q_2 is determined from the relation $\int_0^{q_2} f(q)dq = 1-\alpha$. Finally the SCI is the $(T+\ln\alpha/n, T)$.
Note that if we differentiate with respect the q_2, then the length is still a strictly increasing function, but we can not get $q_2=0$ since $q\in[0, \infty]$ and $q_1<q_2$. Thus we have to differentiate with respect to q_1.

2.3 The case where a SCI does not always exist

The SCI does not always exist even when the distribution of the pivotal quantity f(q) is symmetric. At this point it is worth to comment and make widely known two examples given by Kirmani (1990).

Example 2.3 Let X have the density $f(x, \theta)=|x-\theta|$, $\theta-1<x<\theta+1$, $-\infty<\theta<+\infty$. The quantity $Q=X-\theta$ has the symmetric distribution $f(q)=|q|$ $-1\leq q\leq1$ and is a pivotal one. To find a SCI or a ETCI we use the relation (1.1). At the moment will discuss the case where $-1<q_1<0<q_2<1$. The cases $0<q_1<q_2<1$ and $-1<q_1<q_2<0$ give us CIs whose level of significance is less than 50% since in both cases $½<\alpha<1$.So for the case $-1<q_1<0<q_2<1$ we have: $P(x-q_2\leq\theta\leq x-q_1)=1-\alpha$ and the interval for θ has length $l=q_2-q_1$. Minimizing this length subject to (1.1) gives $f(q_1)=f(q_2)$ and hence $-q_1=q_2$. From that we get that $\left.\frac{d^2l}{dq_1^2}\right|_{q_1=-q_2} < 0$ and hence l(q) can not be minimized (actually is maximized) which means that a SCI does not exist in this case. On the contrary an ETCI exists and has the form $[x-(1-\alpha)^{1/2}, x+(1-\alpha)^{1/2}]$.
At this point we have to say that the proposition 2.1 can not be applied since the density f(q) it is not unimodal.

If we want a SCI or an ETCI for theoretical reasons and not for practical use, we can work out the case $0<q_1<q_2<1$. In this case f(q) is strictly increasing and making use of proposition 2.3 we get that the SCI is of the form $[x-1, x-(2\alpha-1)^{1/2}]$ whereas the ETCI has the form $[x-(1-\alpha)^{1/2}, x+\alpha^{1/2}]$.

Example 2.4 Let X have density $f(x,\theta)=(1/2\theta)e^{-|x|/\theta}$, $-\infty<x<+\infty$, $\theta>0$. The quantity $Q=X/\theta$ is a pivotal quantity with density $f(q)=.5e^{-|q|}$, $-\infty<q<+\infty$. As in the previous example the most interesting case is the case where $-\infty<q_1<0<q_2<+\infty$. The CI we get, based on the previous pivotal quantity, has the form $[\max(x/q_1, x/q_2), +\infty]$. Obviously its length l equals to infinity (l=∞). This implies that there is no meaning to search for a SCI. On the contrary an ETCI can easily be obtained and has the form $[\max(x/\ln\alpha, -x/\ln\alpha), +\infty]$.

Let's now consider the quantity $Q^*=2|x|/\theta$. It can be shown that it is a pivotal quantity with p.d.f. $f(q^*)=.5e^{-q^*/2}$, $q^*>0$. The CI based on this quantity takes the form$\left(\frac{2|x|}{q_2}, \frac{2|x|}{q_1}\right)$. Since f(q*) is decreasing the q_1 and q_2 will be determined from the relation $q_1^2 f(q_1) = q_2^2 f(q_2)$ (see proposition 2.3).

2.4 Bayes approach

Although some textbooks in Mathematical Statistics discuss Bayes confidence intervals (BCI), the concept of a Bayes shortest confidence interval (BSCI) commands little or no attention. The term is mentioned in Rohatgi and Saleh (2001), Casella and Berger (2002), Beaumont (1980), Mood et all (1974) and Silvey (1975). However, neither text offers any further discussion of the topic.

Let X be a r.v having a density $f(x|\theta)$. Suppose that $\pi(\theta)$ is a prior distribution of θ and $\pi(\theta|x)$ is the posterior distribution corresponding to $f(x|\theta)$ and $\pi(\theta)$. Given $\pi(\theta|x)$ the $100(1-\alpha)\%$ BCI for θ is defined by

$$P(q_1<(\theta|x)<q_2)=1-\alpha \quad \text{or} \quad \int_{q_1}^{q_2} \pi(\theta \mid x)d\theta = 1-\alpha . \qquad (2.9)$$

Hence, in order to obtain a BSCI for θ, we need to choose q_1, q_2 such that the length

$$l=q_2-q_1 \qquad (2.10)$$

is minimum under the condition (2.9). In the case where $\pi(\theta|x)$ is symmetric and unimodal then q_1 and q_2 can be determined from the relation $\pi(q_1|x)=\pi(q_2|x)$. In a different case we have to exam the monotonicity of $l(q)$. In the last case the proposition 2.3 can be used without any restriction since the form of l is always of the form (q_2-q_1). In the Bayes approach θ is a r.v. and in general the posterior probability $\pi(\theta|x)$ can be considered as a pivotal quantity, in the sense that it is a function of θ and x has some "known" distribution. After that we can state the following proposition.

Proposition 2.4 If $\pi(\theta|x)$ is the posterior p.d.f. of $\theta|x$, then for the BCI of θ of the form (2.9) and length (2.10) we have that:

i) if $\pi(\theta|x)$ is a strictly increasing p.d.f on $[k_1, k_2]$, $k_i \in R$ (i=1, 2), then $l(q)$ is strictly decreasing on $[k_1, k_2]$.

ii) if $\pi(\theta|x)$ is a strictly decreasing p.d.f on $[k_1, k_2]$, $k_i \in R$ (i=1, 2), then $l(q)$ is strictly increasing on $[k_1, k_2]$.

Those results, as in the classical case, make easier the determination of q_1 and q_2.

Remarks: (i) In the present case theorem 9.3.2 of Casella and Berger is valid without any comment (like those made for the classical case) because the length l is always of the form q_2-q_1. (See also Casella and Berger corollary 9.3.10).

(ii) A BETCI can be defined from the relations

$$\int_{-\infty}^{q_1} \pi(\theta \mid x)d\theta = \alpha/2 \quad \text{and} \quad \int_{q_2}^{+\infty} \pi(\theta \mid x)d\theta = \alpha/2 \ .$$

In this case proposition 2.1 is always true , i. e. if $\pi(\theta|x)$ is symmetric and unimodal then $l_{BSCI}=l_{BETCI}$. The comments made for a similar remark in the classical case are still true.

(iii) Since the determination of q_1 and q_2 is based on the posterior p.d.f., $\pi(\theta|x)$, many authors (see e.g Bickel and Doksum (2001), Casella and Berger (2002)), in order to distinguish between classical and Bayesian confidence sets, they use the term credible sets for the second case.

In many cases the BSCI for a parameter θ has shorter length than the corresponding SCI in the classical case. This it is maybe expected since in the Bayesian approach we have more information about the parameter θ.

We demonstrate the previous discussion with the following examples.

Example 2.5 Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution $N(\theta, 1)$ and let the prior distribution of θ be the $N(0, 1)$. It is well known (see Mood et all 1974, Bickel and Doksum 2001) that the posterior distribution of θ, $\pi(\theta|x)$, is the $N\left(n\bar{X}/(n+1),\ 1/(n+1)\right)$. Since $\pi(\theta|x)$ is symmetric and unimodal, by previous discussion, the q_1 and q_2 will be found from the relation $\pi(q_1|x)=\pi(q_2|x)$ or $\left(q_1 - n\bar{X}/(n+1)\right)^2 = \left(q_2 - n\bar{X}/(n+1)\right)^2$, which implies that $q_1 = 2n\bar{X}/(n+1) - q_2$. Combining it with the relation $P(\theta|x \geq q_2)=\alpha/2$ we get that $q_1 = \frac{n\bar{X}}{n+1} - \frac{1}{\sqrt{n+1}} z_{\alpha/2}$ and hence $q_2 = \frac{n\bar{X}}{n+1} + \frac{1}{\sqrt{n+1}} z_{\alpha/2}$. Thus we get the well known CI which is the shortest. The BETCI are found using the usual relationships. Note that in this case the reverse of proposition 2.1 is also true.

Example 2.6 Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution $U(0, \theta)$ and let the prior distribution of θ be the Pareto with density $\pi(\theta) = k x_0^k/\theta^{k+1}$, $x_0 \leq \theta < \infty$, where x_o and k are known quantities. It can be shown that

$$\pi(\theta \mid x) = \frac{(n+k)(X^*_{(n)})^{n+k}}{\theta^{n+k+1}}, \quad X^*_{(n)} \leq \theta < \infty,$$

where $X^*_{(n)} = \max(X_{(n)}, x_o)$ and $X_{(n)}=\max X_i$. Since $\pi(\theta|x)$ is strictly decreasing on $[X^*_{(n)}, \infty]$, $l(q)$ is strictly increasing on the same interval and hence $q_1 = X^*_{(n)}$ and $q_2 = X^*_{(n)}\alpha^{-\frac{1}{n+k}}$. Thus the BSCI for θ is the $\left(X^*_{(n)},\ X^*_{(n)}\alpha^{-\frac{1}{n+k}}\right)$.

Example 2.7 Let X be a r.v with p.d.f. $f(x, \theta)=e^{-(x-\theta)}$, $-\infty<\theta\leq x<\infty$, and let the prior p.d.f. of θ be the $\pi(\theta)=\theta e^{-\theta}$, $\theta\geq 0$. Here $\pi(\theta|x)=2\theta/x^2$, $0\leq\theta\leq x$. Now, $\pi(\theta|x)$ is strictly increasing on $[0, x]$ which means that $l(q)$ is strictly decreasing on the same interval. Thus the minimum of $l(q)$ occurs at $q_2=x$ and $q_1=x\alpha^{1/2}$, i.e the BSCI is the $(x\alpha^{1/2}, x)$.

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ON LINEAR VOLTERRA DIFFERENCE EQUATIONS WITH INFINITE DELAY

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ABSTRACT. Linear neutral, and especially non-neutral, Volterra difference equations with infinite delay are considered and some new results on the behavior of solutions are established. The results are obtained by the use of appropriate positive roots of the corresponding characteristic equation.

1. PRELIMINARY NOTES

Motivated by the old but significant papers by Driver [3] and Driver, Sasser and Slater [5], a number of relevant papers has recently appeared in the literature. See Frasson and Verduyn Lunel [10], Graef and Qian [11], Kordonis, Niyianni and Philos [16], Kordonis and Philos [18], Kordonis, Philos and Purnaras [21], Philos [27], and Philos and Purnaras [28, 29, 33, 35, 36]. The results in [10, 11, 16, 27, 28, 29, 33, 35] concern the large time behavior of the solutions of several classes of linear autonomous or periodic delay or neutral delay differential equations, while those in [18, 21, 36] are dealing with the behavior of solutions of some linear (neutral or non-neutral) integrodifferential equations with unbounded delay. Note that the method used in [10] is based on resolvent computations and Dunford calculus, while the technique applied in the rest of the papers mentioned above is very simple and is essentially based on elementary calculus. We also notice that the article [10] is very interesting as well as comprehensive.

Along with the work mentioned above for the continuous case, analogous investigations have recently been made for the behavior of the solutions of some classes of linear autonomous or periodic delay or neutral delay difference equations, for the behavior of the solutions of certain linear delay difference equations with continuous variable as well as for the behavior of solutions of a linear Volterra difference equation with infinite delay. See Kordonis and Philos [19], Kordonis, Philos and Purnaras [20], and Philos and Purnaras [30, 31, 32, 34]. For some related results we refer to the papers by De Bruijn [2], Driver, Ladas and Vlahos [4], Györi [12], Norris [25], and Pituk [37, 38].

In [21], Kordonis, Philos and Purnaras obtained some results on the behavior of solutions of linear neutral integrodifferential equations with unbounded delay; the results in [21] extend and improve previous ones given by Kordonis and Philos [18] for the special case of (non-neutral) integrodifferential equations with unbounded delay. In [36], Philos and Purnaras continued the study in [18, 21] and established some further results on the behavior of solutions of linear neutral integrodifferential

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equations with unbounded delay, and, especially, of linear (non-neutral) integrodifferential equations with unbounded delay.

Our purpose in this paper is to give the discrete analogues of the results in [18, 21, 36]. Here, we study the behavior of solutions of linear neutral Volterra difference equations with infinite delay, and, especially, of linear (non-neutral) Volterra difference equations with infinite delay. Our results will be derived by the use of appropriate positive roots of the corresponding characteristic equation. Some of the results of the present paper extend and improve the main results of the authors' previous paper [31].

Neutral, and especially non-neutral, Volterra difference equations with infinite delay have been widely used as mathematical models in mathematical ecology, particularly in population dynamics. Although the bibliography on Volterra integrodifferential equations is quite extended, however there has not yet been analogously much work on the Volterra difference equations. We choose to refer here to the papers by Jaroš and Stavroulakis [13], Kiventidis [15], Kordonis and Philos [17], Ladas, Philos and Sficas [22], and Philos [26] for some results concerning the existence and/or the nonexistence of positive solutions of certain linear Volterra difference equations. Also, for some results on the stability of Volterra difference equations, we typically refer to the papers by Elaydi [6, 7], and Elaydi and Murakami [9] (see, also, the book [8, pp. 239–250]).

For the general background of difference equations, one can refer to the books by Agarwal [1], Elaydi [8], Kelley and Peterson [14], Lakshmikantham and Trigiante [23], Mickens [24], and Sharkovsky, Maistrenko and Romanenko [39].

The paper is organized as follows. Section 2 contains an introduction and some notations. Section 3 is devoted to the statement of the main results (and to some comments on them). The proofs of the main results will be given in Section 4.

2. INTRODUCTION AND NOTATIONS

Throughout the paper, $\mathbf{N}$ stands for the set of all nonnegative integers and $\mathbf{Z}$ stands for the set of all integers. Also, the set of all nonpositive integers will be denoted by $\mathbf{Z}^-$. Moreover, the forward difference operator Δ will be considered to be defined as usual, i.e.

$$\Delta s_n = s_{n+1} - s_n, \quad n \in \mathbf{N}$$

for any sequence $(s_n)_{n\in\mathbf{N}}$ of real numbers.

Consider the linear neutral Volterra difference equation with infinite delay

$$\text{(E)} \qquad \Delta\left(x_n + \sum_{j=-\infty}^{n-1} G_{n-j}x_j\right) = ax_n + \sum_{j=-\infty}^{n-1} K_{n-j}x_j$$

and, especially, the linear (*non-neutral*) Volterra difference equation with infinite delay

$$\text{(E}_0\text{)} \qquad \Delta x_n = ax_n + \sum_{j=-\infty}^{n-1} K_{n-j}x_j,$$

where *a is a real number, and $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are sequences of real numbers. It will be supposed that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is not eventually identically* *zero*. Note that (E_0) is a special case of (E), i.e. the special case where the kernel $(G_n)_{n\in\mathbf{N}-\{0\}}$ is identically zero.

Equation (E) can equivalently be written as follows

$$\Delta\left(x_n+\sum_{j=1}^{\infty}G_jx_{n-j}\right)=ax_n+\sum_{j=1}^{\infty}K_jx_{n-j}$$

and, especially, (E_0) can equivalently be written as

$$\Delta x_n=ax_n+\sum_{j=1}^{\infty}K_jx_{n-j}.$$

By a *solution* of the neutral Volterra difference equation (E) (respectively, of the (non-neutral) Volterra difference equation (E_0)), we mean a sequence $(x_n)_{n\in\mathbf{Z}}$ of real numbers which satisfies (E) (resp., (E_0)) for all $n\in\mathbf{N}$.

In the sequel, by S we will denote the (nonempty) set of all sequences $\phi=(\phi_n)_{n\in\mathbf{Z}^-}$ of real numbers such that, for each $n\in\mathbf{N}$,

$$\Phi_n^G\equiv\sum_{j=-\infty}^{-1}G_{n-j}\phi_j=\sum_{j=n+1}^{\infty}G_j\phi_{n-j}\quad\text{and}\quad\Phi_n^K\equiv\sum_{j=-\infty}^{-1}K_{n-j}\phi_j=\sum_{j=n+1}^{\infty}K_j\phi_{n-j}$$

exist in $\mathbf{R}$. In the special case of (E_0), the set S consists of all sequences $\phi=(\phi_n)_{n\in\mathbf{Z}^-}$ of real numbers such that, for each $n\in\mathbf{N}$, Φ_n^K exists in $\mathbf{R}$.

It is clear that, for any given *initial sequence* $\phi=(\phi_n)_{n\in\mathbf{Z}^-}$ in S, there exists a unique solution $(x_n)_{n\in\mathbf{Z}}$ of the difference equation (E) (resp., of (E_0)) which satisfies the *initial condition*

$$\text{(C)}\qquad x_n=\phi_n\quad\text{for } n\in\mathbf{Z}^-;$$

this solution $(x_n)_{n\in\mathbf{Z}}$ is said to be the solution of the *initial problem* (E)$-$(C) (resp., of the *initial problem* $(E_0)-$(C)) or, more briefly, the solution of (E)$-$(C) (resp., of $(E_0)-$(C)).

With the neutral Volterra difference equation (E) we associate its *characteristic equation*

$$(*)\qquad(\lambda-1)\left(1+\sum_{j=1}^{\infty}\lambda^{-j}G_j\right)=a+\sum_{j=1}^{\infty}\lambda^{-j}K_j,$$

which is obtained by seeking solutions of (E) of the form $x_n=\lambda^n$ for $n\in\mathbf{Z}$, where λ is a positive real number. In particular, the *characteristic equation* of the (non-neutral) Volterra difference equation (E_0) is

$$(*)_0\qquad\lambda-1=a+\sum_{j=1}^{\infty}\lambda^{-j}K_j.$$

The use of a positive root λ_0 of the characteristic equation $(*)$ with the property

$$(\mathrm{P}(\lambda_0))\qquad\sum_{j=1}^{\infty}\lambda_0^{-j}\left(1+\left|1-\frac{1}{\lambda_0}\right|j\right)|G_j|+\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}j\,|K_j|<1$$

plays a crucial role in obtaining the results of this paper. In the special case of the (non-neutral) Volterra difference equation (E_0), the property ($P(\lambda_0)$) (of a positive root λ_0 of the characteristic equation $(*)_0$) takes the form

$$(P_0(\lambda_0)) \qquad \frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}j\,|K_j|<1.$$

In what follows, if λ_0 is a positive root of $(*)$ (resp., of $(*)_0$) with the property ($P(\lambda_0)$) (resp., with the property ($P_0(\lambda_0)$)), we shall denote by $S(\lambda_0)$ the (nonempty) subset of S consisting of all sequences $\phi=(\phi_n)_{n\in\mathbf{Z}^-}$ in S such that $\left(\lambda_0^{-n}\phi_n\right)_{n\in\mathbf{Z}^-}$ is a bounded sequence.

Now, we introduce certain notations which will be used throughout the paper without any further mention. We also give some facts concerning these notations that we shall keep in mind in what follows.

Let λ_0 be a positive root of the characteristic equation $(*)$ with the property ($P(\lambda_0)$). We define

$$\gamma(\lambda_0)=\sum_{j=1}^{\infty}\lambda_0^{-j}\left[1-\left(1-\frac{1}{\lambda_0}\right)j\right]G_j+\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}jK_j$$

and

$$\mu(\lambda_0)=\sum_{j=1}^{\infty}\lambda_0^{-j}\left(1+\left|1-\frac{1}{\lambda_0}\right|j\right)|G_j|+\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}j\,|K_j|\,.$$

Property ($P(\lambda_0)$) together with the hypothesis that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is not eventually identically zero guarantee that

$$0<\mu(\lambda_0)<1.$$

Also, because of $|\gamma(\lambda_0)|\le\mu(\lambda_0)$, we have $-1<\gamma(\lambda_0)<1$, i.e.

$$0<1+\gamma(\lambda_0)<2.$$

In the particular case where $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive and λ_0 is less than or equal to 1, because of the fact that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is not eventually identically zero, the property ($P(\lambda_0)$) can be written as $-1<\gamma(\lambda_0)<0$, i.e.

$$0<1+\gamma(\lambda_0)<1.$$

Furthermore, we set

$$\Theta(\lambda_0)=\frac{[1+\mu(\lambda_0)]^2}{1+\gamma(\lambda_0)}+\mu(\lambda_0).$$

We can easily see that $\Theta(\lambda_0)$ is a real number with

$$\Theta(\lambda_0)>1.$$

Let us consider the special case of the (non-neutral) Volterra difference equation (E_0) and let λ_0 be a positive root of the characteristic equation $(*)_0$ with the property ($P_0(\lambda_0)$). In this case, we define

$$\gamma_0(\lambda_0)=\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}jK_j$$

and

$$\mu_0(\lambda_0) = \frac{1}{\lambda_0}\sum_{j=1}^{\infty} \lambda_0^{-j} j\, |K_j|.$$

From the property $(\mathrm{P}_0(\lambda_0))$ and the hypothesis that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is not eventually identically zero it follows that

$$0 < \mu_0(\lambda_0) < 1.$$

So, since $|\gamma_0(\lambda_0)| \leq \mu_0(\lambda_0)$, we have $-1 < \gamma_0(\lambda_0) < 1$, namely

$$0 < 1 + \gamma_0(\lambda_0) < 2.$$

If $(K_n)_{n\in\mathbf{N}-\{0\}}$ is assumed to be nonpositive, then, by the fact that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is not eventually identically zero, the property $(\mathrm{P}_0(\lambda_0))$ is equivalent to $-1 < \gamma_0(\lambda_0) < 0$, i.e.

$$0 < 1 + \gamma_0(\lambda_0) < 1.$$

Furthermore, we put

$$\Theta_0(\lambda_0) = \frac{[1+\mu_0(\lambda_0)]^2}{1+\gamma_0(\lambda_0)} + \mu_0(\lambda_0)$$

and we see that $\Theta_0(\lambda_0)$ is a real number with

$$\Theta_0(\lambda_0) > 1.$$

We notice that, in the special case of (E_0), the constants $\gamma(\lambda_0)$, $\mu(\lambda_0)$ and $\Theta(\lambda_0)$, which are defined in the general case of (E), are equal to $\gamma_0(\lambda_0)$, $\mu_0(\lambda_0)$ and $\Theta_0(\lambda_0)$, respectively.

Next, consider again a positive root λ_0 of the characteristic equation $(*)$ with the property $(\mathrm{P}(\lambda_0))$, and let $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ be an initial sequence in $S(\lambda_0)$. We define

$$\begin{aligned} L(\lambda_0;\phi) &= \phi_0 + \sum_{j=1}^{\infty} G_j \left[\phi_{-j} - \left(1-\frac{1}{\lambda_0}\right)\lambda_0^{-j}\left(\sum_{r=-j}^{-1}\lambda_0^{-r}\phi_r\right)\right] \\ &\quad + \frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left(\sum_{r=-j}^{-1}\lambda_0^{-r}\phi_r\right) \end{aligned}$$

and

$$M(\lambda_0;\phi) = \sup_{n\in\mathbf{Z}^-}\left|\lambda_0^{-n}\phi_n - \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\right|.$$

From the property $(\mathrm{P}(\lambda_0))$ and the definition of $S(\lambda_0)$ it follows that $L(\lambda_0;\phi)$ is a real number. Moreover, by the definition of $S(\lambda_0)$, $M(\lambda_0;\phi)$ is a nonnegative constant.

Let us concentrate on the special case of the equation (E_0) and consider a positive root λ_0 of the characteristic equation $(*)_0$ with the property $(\mathrm{P}_0(\lambda_0))$ and an initial sequence $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in $S(\lambda_0)$. In this special case, we have the constants

$$L_0(\lambda_0;\phi) = \phi_0 + \frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left(\sum_{r=-j}^{-1}\lambda_0^{-r}\phi_r\right)$$

and

$$M_0(\lambda_0;\phi) = \sup_{n\in\mathbf{Z}^-} \left| \lambda_0^{-n}\phi_n - \frac{L_0(\lambda_0;\phi)}{1+\gamma_0(\lambda_0)} \right|$$

instead of the constants $L(\lambda_0;\phi)$ and $M(\lambda_0;\phi)$ considered in the general case of the equation (E). Property ($\mathrm{P}_0(\lambda_0)$) and the definition of $S(\lambda_0)$ guarantee that $L_0(\lambda_0;\phi)$ is a real number, and the definition of $S(\lambda_0)$ ensures that $M_0(\lambda_0;\phi)$ is a nonnegative constant.

Another notation used in the paper is the following one

$$N(\lambda_0;\phi) = \sup_{n\in\mathbf{Z}^-} \left(\lambda_0^{-n} |\phi_n| \right)$$

for each positive root λ_0 of the characteristic equation $(*)$ (resp., $(*)_0$) with the property ($\mathrm{P}(\lambda_0)$) (resp., ($\mathrm{P}_0(\lambda_0)$)) and for any initial sequence $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in $S(\lambda_0)$. Clearly, $N(\lambda_0;\phi)$ is a nonnegative constant.

Furthermore, let λ_0 be a positive root of the characteristic equation $(*)$ with the property ($\mathrm{P}(\lambda_0)$) and λ_1 be a positive root of $(*)$ with $\lambda_1 < \lambda_0$. Let also $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ be an initial sequence in $S(\lambda_0)$. We set

$$U(\lambda_0,\lambda_1;\phi) = \inf_{n\in\mathbf{Z}^-} \left\{ \lambda_1^{-n} \left[\phi_n - \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)} \lambda_0^n \right] \right\}$$

and

$$V(\lambda_0,\lambda_1;\phi) = \sup_{n\in\mathbf{Z}^-} \left\{ \lambda_1^{-n} \left[\phi_n - \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)} \lambda_0^n \right] \right\}.$$

From the definition of $S(\lambda_0)$ and the hypothesis that $\lambda_1 < \lambda_0$ it follows that $U(\lambda_0,\lambda_1;\phi)$ and $V(\lambda_0,\lambda_1;\phi)$ are real constants.

In particular, consider the special case of (E_0). Let λ_0 be a positive root of the characteristic equation $(*)_0$ with the property ($\mathrm{P}_0(\lambda_0)$) and λ_1 be a positive root of $(*)_0$ with $\lambda_1 < \lambda_0$ as well as let $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ be an initial sequence in $S(\lambda_0)$. In this special case, we consider the real constants

$$U_0(\lambda_0,\lambda_1;\phi) = \inf_{n\in\mathbf{Z}^-} \left\{ \lambda_1^{-n} \left[\phi_n - \frac{L_0(\lambda_0;\phi)}{1+\gamma_0(\lambda_0)} \lambda_0^n \right] \right\}$$

and

$$V_0(\lambda_0,\lambda_1;\phi) = \sup_{n\in\mathbf{Z}^-} \left\{ \lambda_1^{-n} \left[\phi_n - \frac{L_0(\lambda_0;\phi)}{1+\gamma_0(\lambda_0)} \lambda_0^n \right] \right\}$$

in place of $U(\lambda_0,\lambda_1;\phi)$ and $V(\lambda_0,\lambda_1;\phi)$ considered in the general case of (E).

Before closing this section, we will give two well-known definitions. The *trivial solution* of (E) (resp., of (E_0)) is said to be *stable* (*at* 0) if, for each $\epsilon > 0$, there exists $\delta \equiv \delta(\epsilon) > 0$ such that, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in S with $\|\phi\| \equiv \sup_{n\in\mathbf{Z}^-} |\phi_n| < \delta$, the solution $(x_n)_{n\in\mathbf{Z}}$ of (E)$-$(C) (resp., of (E_0)$-$(C)) satisfies $|x_n| < \epsilon$ for all $n \in \mathbf{Z}$. Also, the trivial solution of (E) (resp., of (E_0)) is called *asymptotically stable* (*at* 0) if it is stable (at 0) in the above sense and, in addition, there exists $\delta_0 > 0$ such that, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in S with $\|\phi\| < \delta_0$, the solution $(x_n)_{n\in\mathbf{Z}}$ of (E)$-$(C) (resp., of (E_0)$-$(C)) satisfies $\lim_{n\to\infty} x_n = 0$. Moreover, the trivial solution of (E) (resp., of (E_0)) is called *exponentially stable* (*at* 0) if there exist positive constants Λ and $\eta < 1$ such that, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in S with $\|\phi\| < \infty$, the solution

$(x_n)_{n\in\mathbf{Z}}$ of (E)−(C) (resp., of (E_0)−(C)) satisfies $|x_n| \le \Lambda\eta^n \|\phi\|$ for all $n \in \mathbf{N}$ (see Elaydi and Murakami [9]).

3. STATEMENT OF THE MAIN RESULTS

Our first main result is Theorem 1 below, which establishes a useful inequality for solutions of the neutral Volterra difference equation (E). The application of Theorem 1 to the special case of the (non-neutral) Volterra difference equation (E_0) leads to Theorem 2 below.

Theorem 1. *Let λ_0 be a positive root of the characteristic equation* $(*)$ *with the property* $(\mathrm{P}(\lambda_0))$. *Then, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in $S(\lambda_0)$, the solution $(x_n)_{n\in\mathbf{Z}}$ of* (E)−(C) *satisfies*

$$\left|\lambda_0^{-n}x_n - \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\right| \le \mu(\lambda_0)M(\lambda_0;\phi) \quad \textit{for all } n \in \mathbf{N}.$$

Theorem 2. *Let λ_0 be a positive root of the characteristic equation* $(*)_0$ *with the property* $(\mathrm{P}_0(\lambda_0))$. *Then, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in $S(\lambda_0)$, the solution $(x_n)_{n\in\mathbf{Z}}$ of* (E_0)−(C) *satisfies*

$$\left|\lambda_0^{-n}x_n - \frac{L_0(\lambda_0;\phi)}{1+\gamma_0(\lambda_0)}\right| \le \mu_0(\lambda_0)M_0(\lambda_0;\phi) \quad \textit{for all } n \in \mathbf{N}.$$

Theorem 3 below provides an estimate of solutions of the neutral Volterra difference equation (E) that leads to a stability criterion for the *trivial solution* of (E). By applying Theorem 3 to the special case of the (non-neutral) Volterra difference equation (E_0), one can be led to the subsequent theorem, i.e. Theorem 4.

Theorem 3. *Let λ_0 be a positive root of the characteristic equation* $(*)$ *with the property* $(\mathrm{P}(\lambda_0))$. *Then, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in $S(\lambda_0)$, the solution $(x_n)_{n\in\mathbf{Z}}$ of* (E)−(C) *satisfies*

$$|x_n| \le \Theta(\lambda_0)N(\lambda_0;\phi)\lambda_0^n \quad \textit{for all } n \in \mathbf{N}.$$

Moreover, the trivial solution of (E) *is stable* (*at* 0) *if $\lambda_0 = 1$ and it is asymptotically stable* (*at* 0) *if $\lambda_0 < 1$. In addition, the trivial solution of* (E) *is exponentially stable* (*at* 0) *if $\lambda_0 < 1$.*

Theorem 4. *Let λ_0 be a positive root of the characteristic equation* $(*)_0$ *with the property* $(\mathrm{P}_0(\lambda_0))$. *Then, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in $S(\lambda_0)$, the solution $(x_n)_{n\in\mathbf{Z}}$ of* (E_0)−(C) *satisfies*

$$|x_n| \le \Theta_0(\lambda_0)N(\lambda_0;\phi)\lambda_0^n \quad \textit{for all } n \in \mathbf{N}.$$

Moreover, the trivial solution of (E_0) *is stable* (*at* 0) *if $\lambda_0 = 1$ and it is asymptotically stable* (*at* 0) *if $\lambda_0 < 1$. In addition, the trivial solution of* (E_0) *is exponentially stable* (*at* 0) *if $\lambda_0 < 1$.*

It must be noted that Theorems 2 and 4 for the (non-neutral) Volterra difference equation (E_0) can be considered as substiantally improved versions of the main results of the previous authors' paper [31]. One can easily see the connection between Theorems 2 and 4, and the main results in [31].

The following lemma, i.e. Lemma 1, gives sufficient conditions for the characteristic equation $(*)$ to have a (unique) root λ_0 with the property $(\mathrm{P}(\lambda_0))$. The specialization of Lemma 1 to the special case of the characteristic equation $(*)_0$ is formulated below as Lemma 2. We notice that Lemma 2 has been previously proved in the authors' paper [31].

Lemma 1. *Assume that there exists a positive real number* γ *such that*

$$(\mathrm{H}_1) \qquad \sum_{j=1}^{\infty} \gamma^{-j} |G_j| < \infty \quad \textit{and} \quad \sum_{j=1}^{\infty} \gamma^{-j} |K_j| < \infty,$$

$$(\mathrm{H}_2) \qquad (1-\gamma)\sum_{j=1}^{\infty} \gamma^{-j} G_j + \sum_{j=1}^{\infty} \gamma^{-j} K_j > \gamma - 1 - a$$

and

$$(\mathrm{H}_3) \qquad \sum_{j=1}^{\infty} \gamma^{-j} \left[1 + \left(1 + \frac{1}{\gamma}\right) j\right] |G_j| + \frac{1}{\gamma}\sum_{j=1}^{\infty} \gamma^{-j} j\, |K_j| \leq 1.$$

Then, in the interval (γ, ∞), *the characteristic equation* $(*)$ *admits a unique root* λ_0; *this root has the property* $(\mathrm{P}(\lambda_0))$.

Lemma 2. *Assume that there exists a positive real number* γ *such that*

$$(\mathrm{H}_1)_0 \qquad \sum_{j=1}^{\infty} \gamma^{-j} |K_j| < \infty,$$

$$(\mathrm{H}_2)_0 \qquad \sum_{j=1}^{\infty} \gamma^{-j} K_j > \gamma - 1 - a$$

and

$$(\mathrm{H}_3)_0 \qquad \frac{1}{\gamma}\sum_{j=1}^{\infty} \gamma^{-j} j\, |K_j| \leq 1.$$

Then, in the interval (γ, ∞), *the characteristic equation* $(*)_0$ *admits a unique root* λ_0; *this root has the property* $(\mathrm{P}_0(\lambda_0))$.

Theorem 5 and Corollary 1 below concern the behavior of solutions of the neutral Volterra difference equation (E), while Theorem 6 and Corollary 2 below are dealing with the behavior of solutions of the (non-neutral) Volterra difference equation (E_0).

Theorem 5. *Suppose that* $(G_n)_{n\in\mathbf{N}-\{0\}}$ *and* $(K_n)_{n\in\mathbf{N}-\{0\}}$ *are nonpositive. Let* λ_0 *be a positive root of the characteristic equation* $(*)$ *with* $\lambda_0 \leq 1$ *and with the property* $(\mathrm{P}(\lambda_0))$. *Let also* λ_1 *be a positive root of* $(*)$ *with* $\lambda_1 < \lambda_0$. *Then, for any* $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ *in* $S(\lambda_0)$, *the solution* $(x_n)_{n\in\mathbf{Z}}$ *of* (E)–(C) *satisfies*

$$U(\lambda_0, \lambda_1; \phi) \leq \lambda_1^{-n}\left[x_n - \frac{L(\lambda_0; \phi)}{1+\gamma(\lambda_0)}\lambda_0^n\right] \leq V(\lambda_0, \lambda_1; \phi) \quad \textit{for all } n \in \mathbf{N}.$$

We immediately observe that the double inequality in the conclusion of Theorem 5 can equivalently be written as follows

$$U(\lambda_0, \lambda_1; \phi) \left(\frac{\lambda_1}{\lambda_0}\right)^n \leq \lambda_0^{-n} x_n - \frac{L(\lambda_0; \phi)}{1 + \gamma(\lambda_0)} \leq V(\lambda_0, \lambda_1; \phi) \left(\frac{\lambda_1}{\lambda_0}\right)^n \quad \text{for } n \in \mathbf{N}.$$

Consequently, since $\lambda_1 < \lambda_0$, we obtain

$$\lim_{n\to\infty} \left(\lambda_0^{-n} x_n\right) = \frac{L(\lambda_0; \phi)}{1 + \gamma(\lambda_0)},$$

which establishes the following corollary.

Corollary 1. *Suppose that $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive. Let λ_0 be a positive root of the characteristic equation $(*)$ with $\lambda_0 \leq 1$ and with the property $(\mathrm{P}(\lambda_0))$. Assume that $(*)$ has another positive root less than λ_0. Then, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in $S(\lambda_0)$, the solution $(x_n)_{n\in\mathbf{Z}}$ of $(\mathrm{E})-(\mathrm{C})$ satisfies*

$$\lim_{n\to\infty} \left(\lambda_0^{-n} x_n\right) = \frac{L(\lambda_0; \phi)}{1 + \gamma(\lambda_0)}.$$

Theorem 6. *Suppose that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive. Let λ_0 be a positive root of the characteristic equation $(*)_0$ with the property $(\mathrm{P}_0(\lambda_0))$. Let also λ_1 be a positive root of $(*)_0$ with $\lambda_1 < \lambda_0$. Then, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in $S(\lambda_0)$, the solution $(x_n)_{n\in\mathbf{Z}}$ of $(\mathrm{E}_0)-(\mathrm{C})$ satisfies*

$$U_0(\lambda_0, \lambda_1; \phi) \leq \lambda_1^{-n} \left[x_n - \frac{L_0(\lambda_0; \phi)}{1 + \gamma_0(\lambda_0)} \lambda_0^n\right] \leq V_0(\lambda_0, \lambda_1; \phi) \quad \textit{for all } n \in \mathbf{N}.$$

We see that the double inequality in the conclusion of Theorem 6 is equivalently written as

$$U_0(\lambda_0, \lambda_1; \phi) \left(\frac{\lambda_1}{\lambda_0}\right)^n \leq \lambda_0^{-n} x_n - \frac{L_0(\lambda_0; \phi)}{1 + \gamma_0(\lambda_0)} \leq V_0(\lambda_0, \lambda_1; \phi) \left(\frac{\lambda_1}{\lambda_0}\right)^n \quad \text{for } n \in \mathbf{N}.$$

So, as $\lambda_1 < \lambda_0$, we have

$$\lim_{n\to\infty} \left(\lambda_0^{-n} x_n\right) = \frac{L_0(\lambda_0; \phi)}{1 + \gamma_0(\lambda_0)}.$$

This proves the following corollary.

Corollary 2. *Suppose that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive. Let λ_0 be a positive root of the characteristic equation $(*)_0$ with the property $(\mathrm{P}_0(\lambda_0))$. Assume that $(*)_0$ has another positive root less than λ_0. Then, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in $S(\lambda_0)$, the solution $(x_n)_{n\in\mathbf{Z}}$ of $(\mathrm{E}_0)-(\mathrm{C})$ satisfies*

$$\lim_{n\to\infty} \left(\lambda_0^{-n} x_n\right) = \frac{L_0(\lambda_0; \phi)}{1 + \gamma_0(\lambda_0)}.$$

Now, we state two propositions (Propositions 1 and 2) as well as two lemmas (Lemmas 3 and 4). Proposition 1 and Lemma 3 give some useful information about the positive roots of the characteristic equation $(*)$, while Proposition 2 and Lemma

4 are concerned with the special case of the positive roots of the characteristic equation $(*)_0$.

Proposition 1. *Suppose that $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive. Let λ_0 be a positive root of the characteristic equation $(*)$ with $\lambda_0 \le 1$. If there exists another positive root λ_1 of $(*)$ with $\lambda_1 < \lambda_0$ such that*

$$(\mathrm{Q}(\lambda_1)) \qquad \sum_{j=1}^{\infty} \lambda_1^{-j} j\,|G_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda_1^{-j} j\,|K_j| < \infty,$$

then λ_0 has the property $(\mathrm{P}(\lambda_0))$.

Proposition 2. *Suppose that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive. Let λ_0 be a positive root of the characteristic equation $(*)_0$. If there exists another positive root λ_1 of $(*)_0$ with $\lambda_1 < \lambda_0$ such that*

$$(\mathrm{Q}_0(\lambda_1)) \qquad \sum_{j=1}^{\infty} \lambda_1^{-j} j\,|K_j| < \infty,$$

then λ_0 has the property $(\mathrm{P}_0(\lambda_0))$.

Lemma 3. *Suppose that $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive.*
(I) *If $a = 0$, then $\lambda = 1$ is not a root of the characteristic equation $(*)$.*
(II) *Assume that $a = 0$ and that*

$$(\mathrm{H}_4) \qquad \sum_{j=1}^{\infty} |G_j| \le 1.$$

Then, in the interval $(1, \infty)$, the characteristic equation $(*)$ has no roots.
(III) *Assume that*

$$(\mathrm{H}_5) \qquad \sum_{j=1}^{\infty} j\,|G_j| < \infty,$$

$$(\mathrm{H}_6) \qquad \sum_{j=1}^{\infty} |G_j| + \sum_{j=1}^{\infty} j\,|K_j| \le 1$$

and

$$(\mathrm{H}_7) \qquad \sum_{j=1}^{\infty} |K_j| \ge a.$$

Then, in the interval $(1, \infty)$, the characteristic equation $(*)$ has no roots.
(IV) *Assume that (H_7) holds, and let there exist a positive real number γ with $\gamma < 1$ and $\gamma < a + 1$ so that*

$$(\mathrm{H}_8) \qquad \sum_{j=1}^{\infty} \gamma^{-j} j\,|G_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \gamma^{-j} j\,|K_j| < \infty$$

and

$$(\mathrm{H}_9) \qquad (1-\gamma)\sum_{j=1}^{\infty} \gamma^{-j}\,|G_j| + \sum_{j=1}^{\infty} \gamma^{-j}\,|K_j| > a + 1 - \gamma.$$

Moreover, assume that there exists a real number δ *with* $\delta > 0$ *and* $a < \delta < a+1-\gamma$ *such that*

$$(\mathrm{H}_{10}) \qquad (\delta - a)\sum_{j=1}^{\infty}(a+1-\delta)^{-j}\,|G_j| + \sum_{j=1}^{\infty}(a+1-\delta)^{-j}\,|K_j| < \delta.$$

Then: (i) $\lambda = a+1-\delta$ *is not a root of the characteristic equation* $(*)$. (ii) $\lambda = \gamma$ *is not a root of* $(*)$. (iii) *In the interval* $(a+1-\delta, 1]$, $(*)$ *has a unique root.* (iv) *In the interval* $(\gamma, a+1-\delta)$, $(*)$ *has a unique root.* (*Note: We have* $\delta > 0$ *and* $\gamma < a+1-\delta < 1$.)

Lemma 4. *Suppose that* $(K_n)_{n\in\mathbf{N}-\{0\}}$ *is nonpositive.*

(I) $a > -1$ *is a necessary condition for the characteristic equation* $(*)_0$ *to have at least one positive root.*

(II) *The characteristic equation* $(*)_0$ *has no positive roots greater than or equal to* $a+1$.

(III) *Let* $a > -1$ *and let there exist a positive real number* γ *with* $\gamma < a+1$ *so that*

$$(\mathrm{H}_8)_0 \qquad \sum_{j=1}^{\infty}\gamma^{-j} j\,|K_j| < \infty$$

and

$$(\mathrm{H}_9)_0 \qquad \sum_{j=1}^{\infty}\gamma^{-j}\,|K_j| > a+1-\gamma.$$

Moreover, assume that there exists a real number δ *with* $0 < \delta < a+1-\gamma$ *such that*

$$(\mathrm{H}_{10})_0 \qquad \sum_{j=1}^{\infty}(a+1-\delta)^{-j}\,|K_j| < \delta.$$

Then: (i) $\lambda = a+1-\delta$ *is not a root of the characteristic equation* $(*)_0$. (ii) $\lambda = \gamma$ *is not a root of* $(*)_0$. (iii) *In the interval* $(a+1-\delta, a+1)$, $(*)_0$ *has a unique root.* (iv) *In the interval* $(\gamma, a+1-\delta)$, $(*)_0$ *has a unique root.* (*Note: We have* $\gamma < a+1-\delta < a+1$.)

It is an open problem to examine if Theorem 5, Corollary 1 and Proposition 1 remain valid without the restriction that the root λ_0 of the characteristic equation $(*)$ satisfies $\lambda_0 \le 1$. Such a restriction is not a necessity in the non-neutral case (i.e., in Theorem 6, Corollary 2 and Proposition 2).

Our main results can be extended to the more general case of the linear neutral Volterra-delay difference equation with infinite delay

$$\Delta\left(x_n + \sum_{i=1}^{\infty} c_i x_{n-\sigma_i} + \sum_{j=-\infty}^{n-1} G_{n-j}x_j\right) = a x_n + \sum_{i=1}^{\infty} b_i x_{n-\tau_i} + \sum_{j=-\infty}^{n-1} K_{n-j}x_j$$

and, especially, of the linear neutral Volterra-delay difference equation with infinite delay

$$\Delta x_n = a x_n + \sum_{i=1}^{\infty} b_i x_{n-\tau_i} + \sum_{j=-\infty}^{n-1} K_{n-j} x_j,$$

where c_i and b_i $(i = 1, 2, ...)$ are real numbers, and σ_i and τ_i $(i = 1, 2, ...)$ are positive integers with $\sigma_{i_1} \neq \sigma_{i_2}$ and $\tau_{i_1} \neq \tau_{i_2}$ $(i_1, i_2 = 1, 2, ...;\ i_1 \neq i_2)$.

The neutral Volterra difference equation with infinite delay (E) can be considered as the discrete version of the neutral Volterra integrodifferential equation with unbounded delay

$$(\widehat{\mathrm{E}}) \qquad \left[x(t) + \int_{-\infty}^{t} G(t-s)x(s)ds\right]' = ax(t) + \int_{-\infty}^{t} K(t-s)x(s)ds,$$

where a is a real number, G and K are continuous real-valued functions on the interval $[0, \infty)$, and K is assumed to be not eventually identically zero. In particular, the (non-neutral) Volterra difference equation with infinite delay (E_0) can be viewed as the discrete version of the (non-neutral) Volterra integrodifferential equation with unbounded delay

$$(\widehat{\mathrm{E}}_0) \qquad x'(t) = ax(t) + \int_{-\infty}^{t} K(t-s)x(s)ds.$$

The results obtained in this paper should be looked upon as the discrete analogues of the ones given by Kordonis and Philos [18], Kordonis, Philos and Purnaras [21], and Philos and Purnaras [36], for the neutral Volterra integrodifferential equation with unbounded delay ($\widehat{\mathrm{E}}$) and, especially, for the (non-neutral) Volterra integrodifferential equation with unbounded delay ($\widehat{\mathrm{E}}_0$).

4. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1. Let $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ be an initial sequence in $S(\lambda_0)$, and $(x_n)_{n\in\mathbf{Z}}$ be the solution of (E)$-$(C).

Define

$$y_n = \lambda_0^{-n} x_n \quad \text{for } n \in \mathbf{Z}.$$

Then, for each $n \in \mathbf{N}$, we obtain

$$\begin{aligned} & \Delta\left(x_n + \sum_{j=-\infty}^{n-1} G_{n-j}x_j\right) - ax_n - \sum_{j=-\infty}^{n-1} K_{n-j}x_j \\ \equiv\ & \Delta\left(x_n + \sum_{j=1}^{\infty} G_j x_{n-j}\right) - ax_n - \sum_{j=1}^{\infty} K_j x_{n-j} \\ =\ & \Delta\left[\lambda_0^n\left(y_n + \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j}\right)\right] - a\lambda_0^n y_n - \lambda_0^n \sum_{j=1}^{\infty} \lambda_0^{-j} K_j y_{n-j} \end{aligned}$$

$$
\begin{aligned}
&= \lambda_0^n \left[\lambda_0 \Delta \left(y_n + \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j} \right) + (\lambda_0 - 1) \left(y_n + \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j} \right) \right. \\
&\qquad \left. - a y_n - \sum_{j=1}^{\infty} \lambda_0^{-j} K_j y_{n-j} \right] \\
&= \lambda_0^n \left[\lambda_0 \Delta \left(y_n + \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j} \right) + (\lambda_0 - 1 - a) y_n \right. \\
&\qquad \left. + (\lambda_0 - 1) \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j} - \sum_{j=1}^{\infty} \lambda_0^{-j} K_j y_{n-j} \right] \\
&= \lambda_0^n \left[\lambda_0 \Delta \left(y_n + \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j} \right) - (\lambda_0 - 1) \left(\sum_{j=1}^{\infty} \lambda_0^{-j} G_j \right) y_n \right. \\
&\qquad \left. + \left(\sum_{j=1}^{\infty} \lambda_0^{-j} K_j \right) y_n + (\lambda_0 - 1) \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j} - \sum_{j=1}^{\infty} \lambda_0^{-j} K_j y_{n-j} \right] \\
&= \lambda_0^n \left[\lambda_0 \Delta \left(y_n + \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j} \right) - (\lambda_0 - 1) \sum_{j=1}^{\infty} \lambda_0^{-j} G_j (y_n - y_{n-j}) \right. \\
&\qquad \left. + \sum_{j=1}^{\infty} \lambda_0^{-j} K_j (y_n - y_{n-j}) \right].
\end{aligned}
$$

So, $(x_n)_{n \in \mathbf{Z}}$ satisfies (E) for $n \in \mathbf{N}$ if and only if $(y_n)_{n \in \mathbf{Z}}$ satisfies

$$
\begin{aligned}
\Delta \left(y_n + \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j} \right) &= \left(1 - \frac{1}{\lambda_0} \right) \sum_{j=1}^{\infty} \lambda_0^{-j} G_j (y_n - y_{n-j}) \\
&\quad - \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} K_j (y_n - y_{n-j}) \quad \text{for } n \in \mathbf{N}.
\end{aligned} \tag{4.1}
$$

Moreover, the initial condition (C) can equivalently be written as

$$
y_n = \lambda_0^{-n} \phi_n \quad \text{for } n \in \mathbf{Z}^-. \tag{4.2}
$$

Furthermore, we see that (4.1) becomes

$$
\begin{aligned}
\Delta \left(y_n + \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j} \right) &= \left(1 - \frac{1}{\lambda_0} \right) \sum_{j=1}^{\infty} \lambda_0^{-j} G_j \Delta \left(\sum_{r=n-j}^{n-1} y_r \right) \\
&\quad - \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \Delta \left(\sum_{r=n-j}^{n-1} y_r \right)
\end{aligned}
$$

$$= \Delta\left[\left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left(\sum_{r=n-j}^{n-1}y_r\right)\right.$$
$$\left.-\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left(\sum_{r=n-j}^{n-1}y_r\right)\right]$$

for $n \in \mathbf{N}$. Thus, we have

$$y_n+\sum_{j=1}^{\infty}\lambda_0^{-j}G_jy_{n-j} = \left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left(\sum_{r=n-j}^{n-1}y_r\right)$$
$$-\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left(\sum_{r=n-j}^{n-1}y_r\right)+\Lambda$$

for every $n \in \mathbf{N}$, where

$$\Lambda = \left(y_0+\sum_{j=1}^{\infty}\lambda_0^{-j}G_jy_{-j}\right)-\left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left(\sum_{r=-j}^{-1}y_r\right)$$
$$+\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left(\sum_{r=-j}^{-1}y_r\right).$$

But, by using (4.2) and taking into account the definition of $L(\lambda_0;\phi)$, we can immediately verify that $\Lambda = L(\lambda_0;\phi)$. Hence, (4.1) takes the following equivalent form

$$(4.3)\quad y_n+\sum_{j=1}^{\infty}\lambda_0^{-j}G_jy_{n-j} = \left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left(\sum_{r=n-j}^{n-1}y_r\right)$$
$$-\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left(\sum_{r=n-j}^{n-1}y_r\right)+L(\lambda_0;\phi)\quad \text{for } n\in\mathbf{N}.$$

Next, we set

$$z_n = y_n-\frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\quad \text{for } n\in\mathbf{Z}.$$

Then, we take into account the definition of $\gamma(\lambda_0)$ to show that (4.3) may equivalently be written as follows

$$(4.4)\qquad z_n+\sum_{j=1}^{\infty}\lambda_0^{-j}G_jz_{n-j} = \left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left(\sum_{r=n-j}^{n-1}z_r\right)$$
$$-\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left(\sum_{r=n-j}^{n-1}z_r\right)\quad \text{for } n\in\mathbf{N}.$$

On the other hand, the initial condition (4.2) becomes

$$z_n = \lambda_0^{-n}\phi_n - \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)} \quad \text{for } n \in \mathbf{Z}^-. \tag{4.5}$$

Now, by taking into account the definitions of $(y_n)_{n\in\mathbf{Z}}$ and $(z_n)_{n\in\mathbf{Z}}$, we conclude that what we have to prove is that $(z_n)_{n\in\mathbf{Z}}$ satisfies

$$|z_n| \le \mu(\lambda_0)M(\lambda_0;\phi) \quad \text{for all } n \in \mathbf{N}. \tag{4.6}$$

In the rest of the proof we will establish (4.6). From (4.5) and the definition of $M(\lambda_0;\phi)$ it follows that

$$|z_n| \le M(\lambda_0;\phi) \quad \text{for } n \in \mathbf{Z}^-. \tag{4.7}$$

We will show that

$$|z_n| \le M(\lambda_0;\phi) \quad \text{for all } n \in \mathbf{Z}. \tag{4.8}$$

For this purpose, let us consider an arbitrary real number $\epsilon > 0$. Then (4.7) guarantees that

$$|z_n| < M(\lambda_0;\phi) + \epsilon \quad \text{for } n \in \mathbf{Z}^-. \tag{4.9}$$

We claim that

$$|z_n| < M(\lambda_0;\phi) + \epsilon \quad \text{for every } n \in \mathbf{Z}. \tag{4.10}$$

Otherwise, because of (4.9), there exists an integer $n_0 > 0$ so that

$$|z_n| < M(\lambda_0;\phi) + \epsilon \quad \text{for } n \in \mathbf{Z} \text{ with } n \le n_0 - 1$$

and

$$|z_{n_0}| \ge M(\lambda_0;\phi) + \epsilon.$$

Then, by taking into account the definition of $\mu(\lambda_0)$ and the fact that $0 < \mu(\lambda_0) < 1$, from (4.4) we obtain

$$\begin{aligned}
& M(\lambda_0;\phi) + \epsilon \\
\le\ & |z_{n_0}| \le \sum_{j=1}^{\infty} \lambda_0^{-j}|G_j||z_{n_0-j}| + \left|1 - \frac{1}{\lambda_0}\right| \sum_{j=1}^{\infty} \lambda_0^{-j}|G_j| \left(\sum_{r=n_0-j}^{n_0-1} |z_r|\right) \\
& \qquad + \frac{1}{\lambda_0}\sum_{j=1}^{\infty} \lambda_0^{-j}|K_j| \left(\sum_{r=n_0-j}^{n_0-1} |z_r|\right) \\
\le\ & \left[\sum_{j=1}^{\infty} \lambda_0^{-j}\left(1 + \left|1 - \frac{1}{\lambda_0}\right| j\right)|G_j| + \frac{1}{\lambda_0}\sum_{j=1}^{\infty} \lambda_0^{-j} j\,|K_j|\right][M(\lambda_0;\phi)+\epsilon] \\
\equiv\ & \mu(\lambda_0)\,[M(\lambda_0;\phi)+\epsilon] < M(\lambda_0;\phi)+\epsilon.
\end{aligned}$$

This is a contradiction and consequently our claim is true, i.e., (4.10) holds true. Since (4.10) is fulfilled for all numbers $\epsilon > 0$, we conclude that (4.8) is always satisfied. Finally, using (4.8) and taking again into account the definition of $\mu(\lambda_0)$, from (4.4) we derive, for every $n \in \mathbf{N}$,

$$\begin{aligned} |z_n| &\leq \sum_{j=1}^{\infty} \lambda_0^{-j} |G_j| |z_{n-j}| + \left|1 - \frac{1}{\lambda_0}\right| \sum_{j=1}^{\infty} \lambda_0^{-j} |G_j| \left(\sum_{r=n-j}^{n-1} |z_r| \right) \\ &\qquad + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} |K_j| \left(\sum_{r=n-j}^{n-1} |z_r| \right) \\ &\leq \left[\sum_{j=1}^{\infty} \lambda_0^{-j} \left(1 + \left|1 - \frac{1}{\lambda_0}\right| j \right) |G_j| + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} j |K_j| \right] M(\lambda_0; \phi) \\ &\equiv \mu(\lambda_0) M(\lambda_0; \phi). \end{aligned}$$

Consequently, (4.6) has been proved.

The proof of our theorem is complete.

Proof of Theorem 3. Consider an arbitrary initial sequence $\phi = (\phi_n)_{n \in \mathbf{Z}^-}$ in $S(\lambda_0)$ and let $(x_n)_{n \in \mathbf{Z}}$ be the solution of (E)–(C). Then, by Theorem 1, it holds

$$\left| \lambda_0^{-n} x_n - \frac{L(\lambda_0; \phi)}{1 + \gamma(\lambda_0)} \right| \leq \mu(\lambda_0) M(\lambda_0; \phi) \quad \text{for all } n \in \mathbf{N},$$

which leads to

$$\lambda_0^{-n} |x_n| \leq \frac{|L(\lambda_0; \phi)|}{1 + \gamma(\lambda_0)} + \mu(\lambda_0) M(\lambda_0; \phi) \quad \text{for every } n \in \mathbf{N}.$$

On the other hand, the definitions of $M(\lambda_0; \phi)$ and $N(\lambda_0; \phi)$ give

$$M(\lambda_0; \phi) \leq N(\lambda_0; \phi) + \frac{|L(\lambda_0; \phi)|}{1 + \gamma(\lambda_0)}.$$

Thus, we have

$$\lambda_0^{-n} |x_n| \leq \frac{1 + \mu(\lambda_0)}{1 + \gamma(\lambda_0)} |L(\lambda_0; \phi)| + \mu(\lambda_0) N(\lambda_0; \phi) \quad \text{for } n \in \mathbf{N}. \tag{4.11}$$

But, from the definition of $L(\lambda_0; \phi)$ it follows that

$$\begin{aligned} |L(\lambda_0; \phi)| &\leq |\phi_0| + \sum_{j=1}^{\infty} |G_j| \left[|\phi_{-j}| + \left|1 - \frac{1}{\lambda_0}\right| \lambda_0^{-j} \left(\sum_{r=-j}^{-1} \lambda_0^{-r} |\phi_r| \right) \right] \\ &\qquad + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} |K_j| \left(\sum_{r=-j}^{-1} \lambda_0^{-r} |\phi_r| \right) \\ &= |\phi_0| + \sum_{j=1}^{\infty} \lambda_0^{-j} \left[\lambda_0^{-(-j)} |\phi_{-j}| + \left|1 - \frac{1}{\lambda_0}\right| \left(\sum_{r=-j}^{-1} \lambda_0^{-r} |\phi_r| \right) \right] |G_j| \\ &\qquad + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} \left(\sum_{r=-j}^{-1} \lambda_0^{-r} |\phi_r| \right) |K_j|, \end{aligned}$$

which, because of the definitions of $N(\lambda_0;\phi)$ and $\mu(\lambda_0)$, yields

$$\begin{aligned} &|L(\lambda_0;\phi)| \\ \leq\ & \left[1+\sum_{j=1}^{\infty}\lambda_0^{-j}\left(1+\left|1-\frac{1}{\lambda_0}\right|j\right)|G_j|+\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}j\,|K_j|\right]N(\lambda_0;\phi) \\ =\ & [1+\mu(\lambda_0)]\,N(\lambda_0;\phi). \end{aligned}$$

This together with (4.11) give

$$\lambda_0^{-n}\,|x_n| \leq \left\{\frac{[1+\mu(\lambda_0)]^2}{1+\gamma(\lambda_0)}+\mu(\lambda_0)\right\}N(\lambda_0;\phi) \quad \text{for } n\in\mathbf{N}$$

and hence, by taking into account the definition of $\Theta(\lambda_0)$, we have

$$|x_n| \leq \Theta(\lambda_0)N(\lambda_0;\phi)\lambda_0^n \quad \text{for all } n\in\mathbf{N}. \tag{4.12}$$

We have thus proved the first part of the theorem.

Next, we will establish the stability criterion contained in our theorem. Assume that $\lambda_0 \leq 1$. Consider an arbitrary bounded initial sequence $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in S and define

$$\|\phi\| = \sup_{n\in\mathbf{Z}^-}|\phi_n|.$$

As $\lambda_0 \leq 1$, we immediately see that $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ belongs to $S(\lambda_0)$ and, in addition, that

$$N(\lambda_0;\phi) \leq \|\phi\|. \tag{4.13}$$

The solution $(x_n)_{n\in\mathbf{Z}}$ of (E)$-$(C) satisfies (4.12). By combining (4.12) and (4.13), we obtain

$$|x_n| \leq \Theta(\lambda_0)\,\|\phi\|\,\lambda_0^n \quad \text{for every } n\in\mathbf{N}. \tag{4.14}$$

Since $\lambda_0 \leq 1$, it follows from (4.14) that

$$|x_n| \leq \Theta(\lambda_0)\,\|\phi\| \quad \text{for any } n\in\mathbf{N}.$$

Thus, as $\Theta(\lambda_0) > 1$, we always have

$$|x_n| \leq \Theta(\lambda_0)\,\|\phi\| \quad \text{for all } n\in\mathbf{Z}. \tag{4.15}$$

We have proved that, for any bounded initial sequence $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in S, the solution $(x_n)_{n\in\mathbf{Z}}$ of (E)$-$(C) satisfies (4.14) and (4.15). From (4.15) it follows that the trivial solution of (E) is stable (at 0), provided that $\lambda_0 \leq 1$. Finally, if $\lambda_0 < 1$, then (4.14) ensures that

$$\lim_{n\to\infty} x_n = 0$$

and hence the trivial solution of (E) is asymptotically stable (at 0). Finally, if $\lambda_0 < 1$, then it follows from (4.14) that the trivial solution of (E) is also exponentially stable (at 0).

The proof of the theorem has been finished.

Proof of Lemma 1. Assumption (H_1) guarantees that

$$\sum_{j=1}^{\infty}\lambda^{-j}\,|G_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty}\lambda^{-j}\,|K_j| < \infty, \quad \text{for all } \lambda \geq \gamma$$

and hence the formula

$$F(\lambda) = (\lambda - 1)\left(1 + \sum_{j=1}^{\infty} \lambda^{-j} G_j\right) - a - \sum_{j=1}^{\infty} \lambda^{-j} K_j \quad \text{for } \lambda \geq \gamma$$

defines a continuous real-valued function on the interval $[\gamma, \infty)$. From condition (H_2) it follows that

$$F(\gamma) < 0. \tag{4.16}$$

Furthermore, for each $\lambda \geq \gamma$, we obtain

$$\begin{aligned} \left|\sum_{j=1}^{\infty} \lambda^{-j} G_j\right| &\leq \sum_{j=1}^{\infty} \lambda^{-j} |G_j| = \frac{1}{\lambda} \sum_{j=1}^{\infty} \lambda^{-j+1} |G_j| \\ &\leq \frac{1}{\lambda} \sum_{j=1}^{\infty} \gamma^{-j+1} |G_j| = \frac{\gamma}{\lambda} \sum_{j=1}^{\infty} \gamma^{-j} |G_j| \end{aligned}$$

and consequenly, by the first assumption of (H_1), we have

$$\lim_{\lambda \to \infty} \sum_{j=1}^{\infty} \lambda^{-j} G_j = 0.$$

In a similar way, one can see that

$$\lim_{\lambda \to \infty} \sum_{j=1}^{\infty} \lambda^{-j} K_j = 0.$$

So, we immediately verify that

$$F(\infty) = \infty. \tag{4.17}$$

Now, by using the hypothesis that $(K_n)_{n \in \mathbf{N} - \{0\}}$ is not eventually identically zero as well as condition (H_3), we derive for $\lambda > \gamma$

$$\begin{aligned} F'(\lambda) &= 1 + \sum_{j=1}^{\infty} \lambda^{-j} \left[1 - \left(1 - \frac{1}{\lambda}\right) j\right] G_j + \frac{1}{\lambda} \sum_{j=1}^{\infty} \lambda^{-j} j K_j \\ &\geq 1 - \sum_{j=1}^{\infty} \lambda^{-j} \left[1 + \left(1 + \frac{1}{\lambda}\right) j\right] |G_j| - \frac{1}{\lambda} \sum_{j=1}^{\infty} \lambda^{-j} j |K_j| \\ &> 1 - \sum_{j=1}^{\infty} \gamma^{-j} \left[1 + \left(1 + \frac{1}{\gamma}\right) j\right] |G_j| - \frac{1}{\gamma} \sum_{j=1}^{\infty} \gamma^{-j} j |K_j| \\ &\geq 0, \end{aligned}$$

which means that F is strictly increasing on (γ, ∞). This fact together with (4.16) and (4.17) guarantee that, in the interval (γ, ∞), the equation $F(\lambda) = 0$ (i.e., the characteristic equation $(*)$) has a unique root λ_0. Finally, by using again the hypothesis that $(K_n)_{n \in \mathbf{N} - \{0\}}$ is not eventually identically zero as well as condition

(H_3), we get

$$\begin{aligned}
&\sum_{j=1}^{\infty}\lambda_0^{-j}\left(1+\left|1-\frac{1}{\lambda_0}\right|j\right)|G_j|+\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}j\,|K_j|\\
\leq\ &\sum_{j=1}^{\infty}\lambda_0^{-j}\left[1+\left(1+\frac{1}{\lambda_0}\right)j\right]|G_j|+\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}j\,|K_j|\\
<\ &\sum_{j=1}^{\infty}\gamma^{-j}\left[1+\left(1+\frac{1}{\gamma}\right)j\right]|G_j|+\frac{1}{\gamma}\sum_{j=1}^{\infty}\gamma^{-j}j\,|K_j|\\
\leq\ &1.
\end{aligned}$$

So, the root λ_0 of the characteristic equation $(*)$ has the property $(\mathrm{P}(\lambda_0))$. This completes the proof of the lemma.

Proof of Theorem 5. Let $\phi=(\phi_n)_{n\in\mathbf{Z}^-}$ be an arbitrary initial sequence in $S(\lambda_0)$, and $(x_n)_{n\in\mathbf{Z}}$ be the solution of (E)–(C). Define $(y_n)_{n\in\mathbf{Z}}$ and $(z_n)_{n\in\mathbf{Z}}$ as in the proof of Theorem 1. As it has been shown in the proof of Theorem 1, the fact that $(x_n)_{n\in\mathbf{Z}}$ satisfies (E) for $n\in\mathbf{N}$ is equivalent to the fact that $(z_n)_{n\in\mathbf{Z}}$ satisfies (4.4), while the initial condition (C) becomes (4.5). Furthermore, set

$$w_n=\left(\frac{\lambda_0}{\lambda_1}\right)^n z_n\quad\text{for } n\in\mathbf{Z}.$$

Then it is easy to see that (4.4) can equivalently be written as follows

$$\begin{aligned}
(4.18)\quad w_n+\sum_{j=1}^{\infty}\lambda_1^{-j}G_j w_{n-j}&=\left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left[\sum_{r=n-j}^{n-1}\left(\frac{\lambda_0}{\lambda_1}\right)^{n-r}w_r\right]\\
&\quad-\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left[\sum_{r=n-j}^{n-1}\left(\frac{\lambda_0}{\lambda_1}\right)^{n-r}w_r\right]\quad\text{for } n\in\mathbf{N}.
\end{aligned}$$

Moreover, the initial condition (4.5) is written in the following equivalent form

$$(4.19)\qquad w_n=\lambda_1^{-n}\left[\phi_n-\frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\lambda_0^n\right]\quad\text{for } n\in\mathbf{Z}^-.$$

In view of the definitions of $(y_n)_{n\in\mathbf{Z}}$, $(z_n)_{n\in\mathbf{Z}}$ and $(w_n)_{n\in\mathbf{Z}}$, we have

$$(4.20)\qquad w_n=\lambda_1^{-n}\left[x_n-\frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\lambda_0^n\right]\quad\text{for } n\in\mathbf{Z}.$$

From (4.19) and the definitions of $U(\lambda_0,\lambda_1;\phi)$ and $V(\lambda_0,\lambda_1;\phi)$ it follows that

$$U(\lambda_0,\lambda_1;\phi)=\inf_{s\in\mathbf{Z}^-}w_s\quad\text{and}\quad V(\lambda_0,\lambda_1;\phi)=\sup_{s\in\mathbf{Z}^-}w_s.$$

So, by taking into account (4.20), we immediately conclude that all we have to prove is that $(w_n)_{n\in\mathbf{Z}}$ satisfies

$$\inf_{s\in\mathbf{Z}^-}w_s\leq w_n\leq\sup_{s\in\mathbf{Z}^-}w_s\quad\text{for all } n\in\mathbf{N}.$$

We restrict ourselves to show that

$$(4.21)\qquad w_n\geq\inf_{s\in\mathbf{Z}^-}w_s\quad\text{for every } n\in\mathbf{N}.$$

In a similar manner, one can prove that

$$w_n \leq \sup_{s\in\mathbf{Z}^-} w_s \quad \text{for every } n \in \mathbf{N}.$$

In the rest of the proof we will establish (4.21). To this end, it suffices to show that, for any real number D with $D < \inf_{s\in\mathbf{Z}^-} w_s$, it holds

$$w_n > D \quad \text{for all } n \in \mathbf{N}. \tag{4.22}$$

Let us consider an arbitrary real number D with $D < \inf_{s\in\mathbf{Z}^-} w_s$. Then we obviously have

$$w_n > D \quad \text{for } n \in \mathbf{Z}^-. \tag{4.23}$$

Assume, for the sake of contradiction, that (4.22) fails. Then, because of (4.23), there exists an integer $n_0 > 0$ so that

$$w_n > D \quad \text{for } n \in \mathbf{Z} \text{ with } n \leq n_0 - 1$$

and

$$w_{n_0} \leq D.$$

Hence, by using the hypothesis that $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive and that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is not eventually identically zero and taking into account the assumption that $\lambda_0 \leq 1$, from (4.18) we obtain

$$\begin{aligned}
D \;\geq\; w_{n_0} &= -\sum_{j=1}^{\infty}\lambda_1^{-j}G_j w_{n_0-j} + \left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left[\sum_{r=n_0-j}^{n_0-1}\left(\frac{\lambda_0}{\lambda_1}\right)^{n_0-r} w_r\right] \\
&\quad -\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left[\sum_{r=n_0-j}^{n_0-1}\left(\frac{\lambda_0}{\lambda_1}\right)^{n_0-r} w_r\right] \\
&> D\left\{-\sum_{j=1}^{\infty}\lambda_1^{-j}G_j + \left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left[\sum_{r=n_0-j}^{n_0-1}\left(\frac{\lambda_0}{\lambda_1}\right)^{n_0-r}\right]\right. \\
&\quad \left. -\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left[\sum_{r=n_0-j}^{n_0-1}\left(\frac{\lambda_0}{\lambda_1}\right)^{n_0-r}\right]\right\} \\
&= D\left\{-\sum_{j=1}^{\infty}\lambda_1^{-j}G_j + \left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left[\sum_{\nu=1}^{j}\left(\frac{\lambda_0}{\lambda_1}\right)^{\nu}\right]\right. \\
&\quad \left. -\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left[\sum_{\nu=1}^{j}\left(\frac{\lambda_0}{\lambda_1}\right)^{\nu}\right]\right\} \\
&= D\left\{-\sum_{j=1}^{\infty}\lambda_1^{-j}G_j + \left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\frac{\frac{\lambda_0}{\lambda_1}\left[\left(\frac{\lambda_0}{\lambda_1}\right)^{j}-1\right]}{\frac{\lambda_0}{\lambda_1}-1}\right.
\end{aligned}$$

$$
\left. - \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \frac{\frac{\lambda_0}{\lambda_1}\left[\left(\frac{\lambda_0}{\lambda_1}\right)^j - 1\right]}{\frac{\lambda_0}{\lambda_1} - 1} \right\}
$$

$$
\begin{aligned}
&= \frac{D}{\lambda_0 - \lambda_1} \left\{ -(\lambda_0 - \lambda_1) \sum_{j=1}^{\infty} \lambda_1^{-j} G_j + (\lambda_0 - 1) \sum_{j=1}^{\infty} \lambda_0^{-j} G_j \left[\left(\frac{\lambda_0}{\lambda_1}\right)^j - 1 \right] \right. \\
&\qquad \left. - \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \left[\left(\frac{\lambda_0}{\lambda_1}\right)^j - 1 \right] \right\} \\
&= \frac{D}{\lambda_0 - \lambda_1} \left\{ -[(\lambda_0 - 1) - (\lambda_1 - 1)] \sum_{j=1}^{\infty} \lambda_1^{-j} G_j + (\lambda_0 - 1) \sum_{j=1}^{\infty} \left(\lambda_1^{-j} - \lambda_0^{-j}\right) G_j \right. \\
&\qquad \left. - \sum_{j=1}^{\infty} \left(\lambda_1^{-j} - \lambda_0^{-j}\right) K_j \right\} \\
&= \frac{D}{\lambda_0 - \lambda_1} \left\{ \left[-(\lambda_0 - 1) \sum_{j=1}^{\infty} \lambda_0^{-j} G_j + \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \right] \right. \\
&\qquad \left. - \left[-(\lambda_1 - 1) \sum_{j=1}^{\infty} \lambda_1^{-j} G_j + \sum_{j=1}^{\infty} \lambda_1^{-j} K_j \right] \right\} \\
&= \frac{D}{\lambda_0 - \lambda_1} [(\lambda_0 - 1 - a) - (\lambda_1 - 1 - a)] \\
&= D.
\end{aligned}
$$

This contradiction shows that (4.22) holds true.

The proof of the theorem is now complete.

Proof of Theorem 6. First, let us notice that the main difference between the neutral case and the non-neutral one is the existence (in the neutral case) of the terms

$$
\sum_{j=1}^{\infty} \lambda_1^{-j} G_j w_{n-j}
$$

and

$$
\left(1 - \frac{1}{\lambda_0}\right) \sum_{j=1}^{\infty} \lambda_0^{-j} G_j \left[\sum_{r=n-j}^{n-1} \left(\frac{\lambda_0}{\lambda_1}\right)^{n-r} w_r \right]
$$

in (4.18), which do not appear in the non-neutral case. In the special case of the (non-neutral) Volterra difference equation (E_0), (4.18) becomes

$$
w_n = -\frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \left[\sum_{r=n-j}^{n-1} \left(\frac{\lambda_0}{\lambda_1}\right)^{n-r} w_r \right] \quad \text{for } n \in \mathbf{N}.
$$

The need for assuming, in Theorem 5, that the root λ_0 of the characteristic equation (*) is such that $\lambda_0 \le 1$ is due only to the existence of the second of the above terms in (4.18). After the above observations, we omit the proof of the theorem.

Proof of Proposition 1. Assume that there exists another positive root λ_1 of the characteristic equation (*) with $\lambda_1 < \lambda_0$ such that $(\mathrm{Q}(\lambda_1))$ holds. Clearly,

$$\sum_{j=1}^{\infty} \lambda_1^{-j} G_j \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda_1^{-j} K_j \quad \text{exist in } \mathbf{R}.$$

So, since $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive, we must have

$$\sum_{j=1}^{\infty} \lambda_1^{-j} |G_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda_1^{-j} |K_j| < \infty.$$

(This fact can also be obtained from $(\mathrm{Q}(\lambda_1))$.) This guarantees that

$$\sum_{j=1}^{\infty} \lambda^{-j} |G_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda^{-j} |K_j| < \infty, \qquad \text{for all } \lambda \ge \lambda_1$$

and consequently the formula

$$F(\lambda) = (\lambda - 1)\left(1 + \sum_{j=1}^{\infty} \lambda^{-j} G_j\right) - a - \sum_{j=1}^{\infty} \lambda^{-j} K_j \quad \text{for } \lambda \ge \lambda_1$$

defines a real-valued function F on the interval $[\lambda_1, \infty)$. It follows from assumption $(\mathrm{Q}(\lambda_1))$ that

$$\sum_{j=1}^{\infty} \lambda^{-j} j\,|G_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda^{-j} j\,|K_j| < \infty, \qquad \text{for all } \lambda \ge \lambda_1,$$

which ensures that F is differentiable on $[\lambda_1, \infty)$ with

$$F'(\lambda) = 1 + \sum_{j=1}^{\infty} \lambda^{-j}\left[1 - \left(1 - \frac{1}{\lambda}\right) j\right] G_j + \frac{1}{\lambda}\sum_{j=1}^{\infty} \lambda^{-j} j K_j \quad \text{for } \lambda \ge \lambda_1.$$

Furthermore, by using the hypothesis that $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive and $(K_n)_{n\in\mathbf{N}-\{0\}}$ is not eventually identically zero, it is not difficult to check that F' is strictly increasing on the interval $[\lambda_1, 1]$. (We notice that $0 < \lambda_1 < \lambda_0 \le 1$.)

Now, observe that $F(\lambda_1) = F(\lambda_0) = 0$, and so an application of Rolle's theorem ensures the existence of a real number ξ with $\lambda_1 < \xi < \lambda_0$ so that $F'(\xi) = 0$. Since F' is strictly increasing on $[\xi, 1]$, it follows that F' is always positive on $(\xi, 1]$. Hence, as $\xi < \lambda_0 \le 1$, we conclude, in particular, that $F'(\lambda_0) > 0$, namely that

$$1 + \sum_{j=1}^{\infty} \lambda_0^{-j}\left[1 - \left(1 - \frac{1}{\lambda_0}\right) j\right] G_j + \frac{1}{\lambda_0}\sum_{j=1}^{\infty} \lambda_0^{-j} j K_j > 0.$$

By taking into account the fact that $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive and that $\lambda_0 \le 1$, we see that the last inequality can equivalently be written as follows

$$1 - \sum_{j=1}^{\infty} \lambda_0^{-j}\left(1 + \left|1 - \frac{1}{\lambda_0}\right| j\right) |G_j| - \frac{1}{\lambda_0}\sum_{j=1}^{\infty} \lambda_0^{-j} j\,|K_j| > 0,$$

which means that λ_0 has the property $(\mathrm{P}(\lambda_0))$.
The proof of the proposition is complete.

Proof of Proposition 2. Let λ_1 be a positive root of the characteristic equation $(*)_0$ satisfying $(\mathrm{Q}_0(\lambda_1))$. Then it is obvious that

$$\sum_{j=1}^{\infty} \lambda_1^{-j} K_j \quad \text{exists as a real number}$$

and consequently, as $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive, we have

$$\sum_{j=1}^{\infty} \lambda_1^{-j} |K_j| < \infty.$$

(Note that this fact is also a consequence of $(\mathrm{Q}_0(\lambda_1))$.) Therefore,

$$\sum_{j=1}^{\infty} \lambda^{-j} |K_j| < \infty \quad \text{for all } \lambda \geq \lambda_1$$

and so we can define the real-valued function F_0 on the interval $[\lambda_1, \infty)$ by

$$F_0(\lambda) = \lambda - 1 - a - \sum_{j=1}^{\infty} \lambda^{-j} K_j \quad \text{for } \lambda \geq \lambda_1.$$

Assumption $(\mathrm{Q}_0(\lambda_1))$ guarantees that

$$\sum_{j=1}^{\infty} \lambda^{-j} j\, |K_j| < \infty \quad \text{for all } \lambda \geq \lambda_1$$

and hence F_0 is differentiable on $[\lambda_1, \infty)$ with

$$F_0'(\lambda) = 1 + \frac{1}{\lambda} \sum_{j=1}^{\infty} \lambda^{-j} j K_j \quad \text{for } \lambda \geq \lambda_1.$$

In view of the hypothesis that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive and not eventually identically zero, we can see that F_0' is strictly increasing on the interval $[\lambda_1, \infty)$.

As $F_0(\lambda_1) = F_0(\lambda_0) = 0$, it follows from Rolle's theorem that $F_0'(\xi) = 0$ for some ξ with $\lambda_1 < \xi < \lambda_0$. Since F_0' is strictly increasing on $[\xi, \infty)$, F_0' is positive on the interval (ξ, ∞). This gives, in particular, that $F_0'(\lambda_0) > 0$, i.e.

$$1 + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} j K_j > 0.$$

Finally, by taking into account the fact that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive, we immediately see that λ_0 has the property $(\mathrm{P}_0(\lambda_0))$, which completes our proof.

Proof of Lemma 3. (I). Let us consider the case where $a = 0$. Then the characteristic equation $(*)$ takes the form

$$(*)' \qquad (\lambda - 1)\left(1 + \sum_{j=1}^{\infty} \lambda^{-j} G_j\right) = \sum_{j=1}^{\infty} \lambda^{-j} K_j.$$

From the hypothesis that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive and not eventually identically zero it follows that

$$\sum_{j=1}^{\infty} K_j < 0.$$

Consequently, $\lambda = 1$ cannot be a root of $(*)'$.

(II). Assume that $(*)'$ has a positive root μ with $\mu > 1$. Then

$$(\mu - 1)\left(1 + \sum_{j=1}^{\infty} \mu^{-j} G_j\right) = \sum_{j=1}^{\infty} \mu^{-j} K_j.$$

In view of the fact that $(G_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive and because of the assumption (H_4), we get

$$1 + \sum_{j=1}^{\infty} \mu^{-j} G_j \geq 1 + \sum_{j=1}^{\infty} G_j = 1 - \sum_{j=1}^{\infty} |G_j| \geq 0.$$

Thus,

$$(\mu - 1)\left(1 + \sum_{j=1}^{\infty} \mu^{-j} G_j\right) \geq 0.$$

On the other hand, since $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive and not eventually identically zero, we have

$$\sum_{j=1}^{\infty} \mu^{-j} K_j < 0.$$

We have thus arrived at a contradiction.

(III). A particular consequence of assumption (H_6) is that

$$\sum_{j=1}^{\infty} j\,|K_j| < \infty. \tag{4.24}$$

Assumption (H_5) and (4.24) imply, in particular, that

$$\sum_{j=1}^{\infty} |G_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} |K_j| < \infty.$$

(Note that the first of the these facts can also be obtained from (H_6).) Thus, we can immediately conclude that

$$\sum_{j=1}^{\infty} \lambda^{-j}\,|G_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda^{-j}\,|K_j| < \infty, \quad \text{for all } \lambda \geq 1.$$

So, the formula

$$F(\lambda) = (\lambda - 1)\left(1 + \sum_{j=1}^{\infty} \lambda^{-j} G_j\right) - a - \sum_{j=1}^{\infty} \lambda^{-j} K_j \quad \text{for } \lambda \geq 1$$

introduces a real-valued function F on the interval $[1,\infty)$. From (H_5) and (4.24) it follows that

$$\sum_{j=1}^{\infty}\lambda^{-j}j\,|G_j|<\infty \quad\text{and}\quad \sum_{j=1}^{\infty}\lambda^{-j}j\,|K_j|<\infty, \quad \text{for all } \lambda\geq 1$$

and consequently the function F is differentiable on $[1,\infty)$ with

$$F'(\lambda)=1+\sum_{j=1}^{\infty}\lambda^{-j}\left[1-\left(1-\frac{1}{\lambda}\right)j\right]G_j+\frac{1}{\lambda}\sum_{j=1}^{\infty}\lambda^{-j}jK_j \quad \text{for } \lambda\geq 1.$$

Furthermore, by the hypothesis that $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive and $(K_n)_{n\in\mathbf{N}-\{0\}}$ is not eventually identically zero, we obtain for $\lambda>1$

$$\begin{aligned}
F'(\lambda) &= 1+\sum_{j=1}^{\infty}\lambda^{-j}G_j-\left(1-\frac{1}{\lambda}\right)\sum_{j=1}^{\infty}\lambda^{-j}jG_j+\frac{1}{\lambda}\sum_{j=1}^{\infty}\lambda^{-j}jK_j\\
&= 1-\sum_{j=1}^{\infty}\lambda^{-j}\,|G_j|+\left(1-\frac{1}{\lambda}\right)\sum_{j=1}^{\infty}\lambda^{-j}j\,|G_j|-\frac{1}{\lambda}\sum_{j=1}^{\infty}\lambda^{-j}j\,|K_j|\\
&\geq 1-\sum_{j=1}^{\infty}\lambda^{-j}\,|G_j|-\frac{1}{\lambda}\sum_{j=1}^{\infty}\lambda^{-j}j\,|K_j|\\
&> 1-\sum_{j=1}^{\infty}|G_j|-\sum_{j=1}^{\infty}j\,|K_j|.
\end{aligned}$$

Hence, by assumption (H_6), we find

$$F'(\lambda)>0 \quad \text{for every } \lambda>1.$$

This implies that F is strictly increasing on the interval $(1,\infty)$. Since $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive, assumption (H_7) means that

$$F(1)\geq 0. \tag{4.25}$$

Thus, the characteristic equation (∗) cannot have roots in the interval $(1,\infty)$.

(IV). Assumption (H_7) means that (4.25) is true. Furthermore, assumption (H_8) guarantees, in particular, that

$$\sum_{j=1}^{\infty}\gamma^{-j}\,|G_j|<\infty \quad\text{and}\quad \sum_{j=1}^{\infty}\gamma^{-j}\,|K_j|<\infty$$

and consequently

$$\sum_{j=1}^{\infty}\lambda^{-j}\,|G_j|<\infty \quad\text{and}\quad \sum_{j=1}^{\infty}\lambda^{-j}\,|K_j|<\infty, \quad \text{for all } \lambda\geq\gamma.$$

So, the formula

$$F(\lambda)=(\lambda-1)\left(1+\sum_{j=1}^{\infty}\lambda^{-j}G_j\right)-a-\sum_{j=1}^{\infty}\lambda^{-j}K_j \quad \text{for } \lambda\geq\gamma$$

defines a real-valued function F on the interval $[\gamma, \infty)$. From assumption (H_8) it follows that

$$\sum_{j=1}^{\infty} \lambda^{-j} j\,|G_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda^{-j} j\,|K_j| < \infty, \quad \text{for all } \lambda \geq \gamma,$$

which ensures that the function F is differentiable on $[\gamma, \infty)$ with

$$F'(\lambda) = 1 + \sum_{j=1}^{\infty} \lambda^{-j} \left[1 - \left(1 - \frac{1}{\lambda}\right) j\right] G_j + \frac{1}{\lambda} \sum_{j=1}^{\infty} \lambda^{-j} j K_j \quad \text{for } \lambda \geq \gamma.$$

By using the hypothesis that $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive and $(K_n)_{n\in\mathbf{N}-\{0\}}$ is not eventually identically zero, we can easily verify that F' is strictly increasing on the interval $[\gamma, 1]$. Consequently,

(4.26) $$F \text{ is strictly convex on } [\gamma, 1].$$

Furthermore, we take into account the fact that $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive to conclude that assumption (H_9) means that

(4.27) $$F(\gamma) > 0,$$

while assumption (H_{10}) means that

(4.28) $$F(a + 1 - \delta) < 0.$$

A particular consequence of (4.27) is that $\lambda = \gamma$ is not a root of $(*)$. Similarily, (4.28) guarantees, in particular, that $\lambda = a + 1 - \delta$ is not a root of $(*)$. Moreover, from (4.25), (4.26) and (4.28) it follows that, in the interval $(a + 1 - \delta, 1]$, $(*)$ has a unique root. Finally, (4.26), (4.27) and (4.28) ensure that, in the interval $(\gamma, a + 1 - \delta)$, $(*)$ has also a unique root.

The lemma has now been proved.

Proof of Lemma 4. (I) and (II). Let us assume that the characteristic equation $(*)_0$ admits a positive root μ. Then

$$\mu - 1 - a = \sum_{j=1}^{\infty} \mu^{-j} K_j.$$

Since $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive and not eventually identically zero, we always have

$$\sum_{j=1}^{\infty} \mu^{-j} K_j < 0.$$

So, we must have $\mu - 1 - a < 0$, i.e. $\mu < a + 1$. This shows Part (II). Moreover, it follows that $a + 1 > 0$, namely $a > -1$, and hence Part (I) has been established.

(III). From assumption $(\mathrm{H}_8)_0$ it follows, in particular, that

$$\sum_{j=1}^{\infty} \gamma^{-j}\,|K_j| < \infty,$$

which guarantees that

$$\sum_{j=1}^{\infty} \lambda^{-j}\,|K_j| < \infty \quad \text{for all } \lambda \geq \gamma.$$

Hence, we can define the real-valued function F_0 on $[\gamma, \infty)$ by the formula

$$F_0(\lambda) = \lambda - 1 - a - \sum_{j=1}^{\infty} \lambda^{-j} K_j \quad \text{for } \lambda \geq \gamma.$$

By $(\mathrm{H}_8)_0$, we see that

$$\sum_{j=1}^{\infty} \lambda^{-j} j\, |K_j| < \infty \quad \text{for all } \lambda \geq \gamma$$

and consequently F_0 is differentiable on $[\gamma, \infty)$ with

$$F_0'(\lambda) = 1 + \frac{1}{\lambda} \sum_{j=1}^{\infty} \lambda^{-j} j K_j \quad \text{for } \lambda \geq \gamma.$$

Furthermore, the hypothesis that $(K_n)_{n \in \mathbf{N} - \{0\}}$ is nonpositive and not eventually identically zero ensures that F_0' is strictly increasing on the interval $[\gamma, \infty)$. So,

(4.29) $$F_0 \text{ is strictly convex on } [\gamma, \infty).$$

Now, as $(K_n)_{n \in \mathbf{N} - \{0\}}$ is nonpositive, assumption $(\mathrm{H}_9)_0$ means that

(4.30) $$F_0(\gamma) > 0,$$

while assumption $(\mathrm{H}_{10})_0$ means that

(4.31) $$F_0(a + 1 - \delta) < 0.$$

From (4.30) it follows, in particular, that $\lambda = \gamma$ is not a root of $(*)_0$, while (4.31) ensures, in particular, that $\lambda = a + 1 - \delta$ is not a root of $(*)_0$. Next, by taking into account the fact that $(K_n)_{n \in \mathbf{N} - \{0\}}$ is nonpositive and not eventually identically zero, we see that

(4.32) $$F_0(a + 1) > 0.$$

Because of (4.29), (4.31) and (4.32), we conclude that, in the interval $(a+1-\delta, a+1)$, $(*)_0$ has a unique root. Moreover, (4.29), (4.30) and (4.31) guarantee that, in the interval $(\gamma, a + 1 - \delta)$, $(*)_0$ admits also a unique root.

We have thus completed the proof of our lemma.

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OSCILLATION CRITERIA FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT

Consider the first-order linear delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1}$$

and the second-order linear delay equation

$$x''(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{2}$$

where $p \in C([t_0, \infty), \mathbb{R}^+)$, $\tau \in C([t_0, \infty), \mathbb{R})$, $\tau(t)$ is non-decreasing, $\tau(t) \leq t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$.

The most interesting oscillation criteria for Eq.(1), especially in the case where

$$0 < \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \leq \frac{1}{e} \quad \text{and} \quad \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds < 1,$$

and for Eq. (2) when

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds \leq \frac{1}{e} \quad \text{and} \quad \limsup_{t\to\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds < 1$$

are presented.

[0] *Key Words:* Oscillation; delay differential equations.
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1 Introduction

The problem of establishing sufficient conditions for the oscillation of all solutions to the first-order differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1}$$

and the second-order equation

$$x''(t) + p(t)x(\tau(t)) = 0, \ t \geq t_0, \tag{2}$$

where $p \in C([t_0,\infty), \mathbb{R}^+)$ (here $\mathbb{R}^+ = [0, \infty)$), $\tau \in C([t_0, \infty), \mathbb{R})$, $\tau(t)$ is non-decreasing, $\tau(t) \leq t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$, has been the subject of many investigations. See, for example, [1-65] and the references cited therein.

By a solution of Eq.(1) [resp. Eq.(2)] we understand a continuously differentiable function defined on $[\tau(T_0), \infty)$ for some $T_0 \geq t_0$ and such that Eq.(1) [resp. Eq.(2)] is satisfied for $t \geq T_0$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory.*

In this paper our main purpose is to present the state of the art on the oscillation of all solutions to Eq.(1) especially in the case where

$$0 < \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \leq \frac{1}{e} \text{ and } \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds < 1,$$

and for Eq.(2) when

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds \leq \frac{1}{e} \text{ and } \limsup_{t\to\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds < 1.$$

2 Oscillation Criteria for Eq. (1)

In this section we study the delay equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0. \tag{1}$$

The first systematic study for the oscillation of all solutions to Eq.(1) was made by Myshkis. In 1950 [42] he proved that every solution of Eq.(1) oscillates if

$$\limsup_{t\to\infty}[t - \tau(t)] < \infty \quad \text{and} \quad \liminf_{t\to\infty}[t - \tau(t)] \liminf_{t\to\infty} p(t) > \frac{1}{e}. \tag{C_1}$$

In 1972, Ladas, Lakshmikantham and Papadakis [33] proved that the same conclusion holds if

$$A := \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1. \qquad (C_2)$$

In 1979, Ladas [32] established integral conditions for the oscillation of Eq.(1) with constant delay. Tomaras [54-56] extended this result to Eq.(1) with variable delay. For related results see Ladde [36-38]. The following most general result is due to Koplatadze and Canturija [25].

If

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}, \qquad (C_3)$$

then all solutions of Eq.(1) oscillate; If

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds < \frac{1}{e}, \qquad (N_1)$$

then Eq.(1) has a nonoscillatory solution.

In 1982 Ladas, Sficas and Stavroulakis [35] and in 1984 Fukagai and Kusano [13] established oscillation criteria (of the type of conditions (C_2)and (C_3)) for Eq. (1) with *oscillating* coefficient $p(t)$.

It is obvious that there is a gap between the conditions (C_2) and (C_3) when the limit $\lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds$ does not exist. How to fill this gap is an interesting problem which has been recently investigated by several authors.

In 1988, Erbe and Zhang [12] developed new oscillation criteria by employing the upper bound of the ratio $x(\tau(t))/x(t)$ for possible nonoscillatory solutions $x(t)$ of Eq.(1). Their result says that all the solutions of Eq.(1) are oscillatory, if $0 < \alpha \leq \frac{1}{e}$ and

$$A > 1 - \frac{\alpha^2}{4}. \qquad (C_4)$$

Since then several authors tried to obtain better results by improving the upper bound for $x(\tau(t))/x(t)$.

In 1991, Jian [20] derived the condition

$$A > 1 - \frac{\alpha^2}{2(1-\alpha)}, \qquad (C_5)$$

while in 1992, Yu and Wang [63] and Yu, Wang, Zhang and Qian [64] obtained the condition

$$A > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}. \qquad (C_6)$$

In 1990, Elbert and Stavroulakis [8] and in 1991 Kwong [30], using different techniques, improved (C_4), in the case where $0 < \alpha \leq \frac{1}{e}$, to the conditions

$$A > 1 - (1 - \frac{1}{\sqrt{\lambda_1}})^2 \tag{C_7}$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1}, \tag{C_8}$$

respectively, where λ_1 is the smaller real root of the equation $\lambda = e^{\alpha\lambda}$.

In 1994, Koplatadze and Kvinikadze [26] improved (C_6), while in 1998, Philos and Sficas [45] and in 1999, Zhou and Yu [65] and Jaroš and Stavroulakis [19] derived the conditions

$$A > 1 - \frac{\alpha^2}{2(1-\alpha)} - \frac{\alpha^2}{2}\lambda_1, \tag{C_9}$$

$$A > 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2} - (1 - \frac{1}{\sqrt{\lambda_1}})^2, \tag{C_{10}}$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}, \tag{C_{11}}$$

respectively.

Consider Eq.(1) and assume that $\tau(t)$ is continuously differentiable and that there exists $\theta > 0$ such that $p(\tau(t))\tau'(t) \geq \theta p(t)$ eventually for all t. Under this additional condition, in 2000, Kon, Sficas and Stavroulakis [22] and in 2003, Sficas and Stavroulakis [46] established the conditions

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - \alpha - \sqrt{(1-\alpha)^2 - 4\Theta}}{2} \tag{2.1}$$

and

$$A > \frac{\ln \lambda_1}{\lambda_1} - \frac{1 + \sqrt{1 + 2\theta - 2\theta\lambda_1 M}}{\theta\lambda_1} \tag{2.2}$$

respectively, where

$$\Theta = \frac{e^{\lambda_1\theta\alpha} - \lambda_1\theta\alpha - 1}{(\lambda_1\theta)^2}$$

and

$$M = \frac{1 - \alpha - \sqrt{(1-\alpha)^2 - 4\Theta}}{2}.$$

Remark 2.1 ([22], [46]) Observe that when $\theta = 1$, then $\Theta = \frac{\lambda_1 - \lambda_1\alpha - 1}{\lambda_1{}^2}$, and (2.1) reduces to

$$A > 2\alpha + \frac{2}{\lambda_1} - 1, \qquad (C_{12})$$

while in this case it follows that $M = 1 - \alpha - \frac{1}{\lambda_1}$ and (2.2) reduces to

$$A > \frac{\ln\lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2\alpha\lambda_1}}{\lambda_1}. \qquad (C_{13})$$

In the case where $\alpha = \frac{1}{e}$, then $\lambda_1 = e$, and (C_{13}) leads to

$$A > \frac{\sqrt{7 - 2e}}{e} \approx 0.459987065.$$

It is to be noted that as $\alpha \to 0$, then all the previous conditions $(C_4) - (C_{12})$ reduce to the condition (C_2), i.e.

$$A > 1.$$

However, the condition (C_{13}) leads to

$$A > \sqrt{3} - 1 \approx 0.732,$$

which is an essential improvement. Moreover (C_{13}) improves all the above conditions when $0 < \alpha \leq \frac{1}{e}$ as well. Note that the value of the lower bound on A can not be less than

$$\frac{1}{e} \approx 0.367879441.$$

Thus the aim is to establish a condition which leads to a value *as close as possible to* $\frac{1}{e}$. For illustrative purpose, we give the values of the lower bound on A under these conditions when $\alpha = \frac{1}{e}$.

(C_4):	0.966166179
(C_5):	0.892951367
(C_6):	0.863457014
(C_7):	0.845181878
(C_8):	0.735758882
(C_9):	0.709011646
(C_{10}):	0.708638892
(C_{11}):	0.599215896
(C_{12}):	0.471517764
(C_{13}):	0.459987065

We see that the condition (C_{13}) essentially improves all the known results in the literature.

Example 2.1 ([46]) Consider the delay differential equation

$$x'(t) + px(t - q\sin^2\sqrt{t} - \frac{1}{pe}) = 0,$$

where $p > 0$, $q > 0$ and $pq = 0.46 - \frac{1}{e}$. Then

$$\alpha = \liminf_{t\to\infty} \int_{\tau(t)}^{t} pds = \liminf_{t\to\infty} p(q\sin^2\sqrt{t} + \frac{1}{pe}) = \frac{1}{e}$$

and

$$A = \limsup_{t\to\infty} \int_{\tau(t)}^{t} pds = \limsup_{t\to\infty} p(q\sin^2\sqrt{t} + \frac{1}{pe}) = pq + \frac{1}{e} = 0.46.$$

Thus, according to Remark 2.1, all solutions of this equation oscillate. Observe that none of the conditions (C_4)-(C_{12}) apply to this equation.

Following this historical (and chronological) review we also mention that in the case where

$$\int_{\tau(t)}^{t} p(s)ds \geq \frac{1}{e} \quad \text{and} \quad \lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \frac{1}{e}$$

this problem has been studied in 1995, by Elbert and Stavroulakis [9], by Kozakiewicz [28], Li [40,41] and in 1996, by Domshlak and Stavroulakis [6].

3 Oscillation Criteria for Eq. (2)

In this section we study the second-order delay equation

$$x''(t) + p(t)x(\tau(t)) = 0, \ t \geq t_0, \tag{2}$$

For the case of ordinary differential equations, i.e. when $\tau(t) \equiv t$, the history of the problem began as early as in 1836 by the work of Sturm [47] and was continued in 1893 by A. Kneser [21]. Essential contribution to the subject was made by E. Hille, A. Wintner, Ph. Hartman, W. Leighton, Z. Nehari, and others (see the monograph by C. Swanson [48] and the references cited therein). In particular, in 1948 E. Hille [17] obtained the following well-known oscillation criteria. Let

$$\limsup_{t\to\infty} t \int_{t}^{+\infty} p(s)ds > 1 \tag{3.1}$$

or

$$\liminf_{t\to\infty} t\int_t^{+\infty} p(s)ds > \frac{1}{4}, \tag{3.2}$$

the conditions being assumed to be satisfied if the integral diverges. Then *Eq.(2)* with $\tau(t) \equiv t$ is oscillatory.

For the delay differential equation (2) earlier oscillation results can be found in the monographs by A. Myshkis [43] and S. Norkin [44]. In 1968 P. Waltman [57] and in 1970 J. Bradley [1] proved that (2) is oscillatory if

$$\int^{+\infty} p(t)dt = +\infty.$$

Proceeding in the direction of generalization of Hille's criteria, in 1971 J. Wong [60] showed that if $\tau(t) \geq \alpha t$ for $t \geq 0$ with $0 < \alpha \leq 1$, then the condition

$$\liminf_{t\to\infty} t\int_t^{+\infty} p(s)ds > \frac{1}{4\alpha} \tag{3.3}$$

is sufficient for the oscillation of Eq.(2). In 1973 L. Erbe [10] generalized this condition to

$$\liminf_{t\to\infty} t\int_t^{+\infty} \frac{\tau(s)}{s} p(s)ds > \frac{1}{4} \tag{3.4}$$

without any additional restriction on τ. In 1987 J. Yan [61] obtained some general criteria improving the previous ones.

An oscillation criterion of different type is given in 1986 by R. Koplatadze [23] and in 1988 by J. Wei [59], where it is proved that Eq.(2) is oscillatory if

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds > 1 \tag*{$(C_2)'$}$$

or

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds > \frac{1}{e}. \tag*{$(C_3)'$}$$

The conditions $(C_2)'$ and $(C_3)'$ are analogous to the oscillation conditions

$$A = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1, \tag*{(C_2)}$$

$$\alpha = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e} \tag*{(C_3)}$$

respectively, for the first order delay equation

$$x'(t) + p(t)x(\tau(t)) = 0. \tag{1}$$

The essential difference between (3.3),(3.4) and $(C_2)'$, $(C_3)'$ is that the first two can guarantee oscillation for ordinary differential equations as well, while the last two work only for delay equations. Unlike first order differential equations, where the oscillatory character is due to the delay only, the equation (2) can be oscillatory without any delay at all, i.e., in the case $\tau(t) \equiv t$. Figuratively speaking, two factors contribute to the oscillatory character of Eq.(2): the presence of the delay and the second order nature of the equation. The conditions (3.3), (3.4) and $(C_2)'$, $(C_3)'$ illustrate the role of these factors taken separately.

In what follows it will be assumed that the condition

$$\int^{+\infty} \tau(s)p(s)ds = +\infty \tag{3.5}$$

is fulfilled. As it follows from Lemma 4.1 in [24], this condition is necessary for Eq.(2) to be oscillatory. The study being devoted to the problem of oscillation of Eq.(2), the condition (3.5) does not affect the generality.

In this section oscillation results are obtained for Eq. (2) by reducing it to a first order equation. Since for the latter the oscillation is due solely to the delay, the criteria hold for delay equations only and do not work in the ordinary case.

Theorem 3.1 ([27]) Let (3.5) be fulfilled and the differential inequality

$$x'(t) + \left(\tau(t) + \int_T^{\tau(t)} \xi\tau(\xi)p(\xi)d\xi\right) p(t)x(\tau(t)) \leq 0$$

have no eventually positive solution. Then Eq. (2) is oscillatory.

Theorem 3.1 reduces the question of oscillation of Eq.(2) to that of the absence of eventually positive solutions of the differential inequality

$$x'(t) + \left(\tau(t) + \int_T^{\tau(t)} \xi\tau(\xi)p(\xi)d\xi\right) p(t)x(\tau(t)) \leq 0. \tag{3.6}$$

So oscillation results for first order delay differential equations can be applied since the oscillation of the equation

$$u'(t) + g(t)u(\delta(t)) = 0 \tag{3.7}$$

is equivalent to the absence of eventually positive solutions of the inequality

$$u'(t) + g(t)u(\delta(t)) \leq 0. \tag{3.8}$$

This fact is a simple consequence of the following comparison theorem deriving the oscillation of (3.7) from the oscillation of the equation

$$v'(t) + h(t)v(\sigma(t)) = 0. \tag{3.9}$$

We assume that $g, h : R_+ \to R_+$ are locally integrable, $\delta, \sigma : R_+ \to R$ are continuous, $\delta(t) \leq t$, $\sigma(t) \leq t$ for $t \in R_+$, and $\delta(t) \to +\infty$, $\sigma(t) \to +\infty$ as $t \to +\infty$.

Theorem 3.2 Let

$$g(t) \geq h(t) \quad \text{and} \quad \delta(t) \leq \sigma(t) \text{ for } t \in R_+$$

and let the equation (3.9) be oscillatory. Then (3.7) is also oscillatory.

Corollary 3.1 Let the equation (3.7) be oscillatory. Then the inequality (3.8) has no eventually positive solution.

Turning to applications of Theorem 3.1, we will use it together with the criteria (C_2) and (C_3) to get

Theorem 3.3 ([27]) Let

$$K := \limsup_{t\to\infty} \int_{\tau(t)}^{t} \left(\tau(s) + \int_0^{\tau(s)} \xi\tau(\xi)p(\xi)d\xi \right) p(s)ds > 1, \qquad (C_2)''$$

or

$$k := \liminf_{t\to\infty} \int_{\tau(t)}^{t} \left(\tau(s) + \int_0^{\tau(s)} \xi\tau(\xi)p(\xi)d\xi \right) p(s)ds > \frac{1}{e}. \qquad (C_3)''$$

Then Eq. (2) is oscillatory.

To apply Theorem (3.1) it suffices to note that: (i) (3.5) is fulfilled since otherwise $k = K = 0$; (ii) since $\tau(t) \to +\infty$ as $t \to +\infty$, the relations $(C_2)''$, $(C_3)''$ imply the same relations with 0 changed by any $T \geq 0$.

Remark 3.1 ([27]) Theorem 3.3 improves the criteria $(C_2)'$, $(C_3)'$ by Koplatadze [23] and Wei [59] mantioned above. This is directly seen from $(C_2)''$, $(C_3)''$ and can be easily checked if we take $\tau(t) \equiv t - \tau_0$ and $p(t) \equiv p_0/(t - \tau_0)$ for $t \geq 2\tau_0$, where the constants $\tau_0 > 0$ and $p_0 > 0$ satisfy

$$\tau_0 p_0 < \frac{1}{e}.$$

In this case neither of $(C_2)'$, $(C_3)'$ is applicable for Eq. (2) while both $(C_2)''$, $(C_3)''$ give the positive conclusion about its oscillation. Note also that this is exactly the case where the oscillation is due to the delay since the corresponding equation without delay is nonoscillatory.

Remark 3.2 ([27]) The criteria $(C_2)''$, $(C_3)''$ look like (C_2), (C_3) but there is an essential difference between them pointed out in the introduction. The condition (C_3) is close to the necessary one since according to [25] if $A \leq \frac{1}{e}$, then (3.7) is nonoscillatory. On the other hand, for an oscillatory equation (2) without delay we have $k = K = 0$. Nevertheless, the constant $1/e$ in Theorem 3.3 is also the best possible in the sense that for any $\varepsilon \in (0, 1/e]$ it can not be replaced by $1/e - \varepsilon$ without affecting the validity of the theorem. This is illustrated by the following

Example 3.1 ([27]) Let $\varepsilon \in (0, 1/e]$, $1 - e\varepsilon < \beta < 1$, $\tau(t) \equiv \alpha t$ and $p(t) \equiv \beta(1-\beta)\alpha^{-\beta}t^{-2}$, where $\alpha = e^{\frac{1}{\beta-1}}$. Then $(C_3)''$ is fulfilled with 1/e replaced by $1/e - \varepsilon$. Nevertheless Eq. (2) has a nonoscillatory solution, namely $u(t) \equiv t^{\beta}$. Indeed, denoting $c = \beta(1-\beta)\alpha^{-\beta}$, we see that the expression under the limit sign in $(C_3)''$ is constant and equals $\alpha c \, |\ln \alpha| \, (1 + \alpha c) = (\beta/e)(1 + (\beta(1-\beta))/e) > \beta/e > 1/e - \varepsilon$.

Note that there is a gap between the conditions $(C_2)''$,$(C_3)''$ when $0 \le k \le 1/e$, $k < K$. In the case of first order equations the conditions (C_4) - (C_{13}) attempt to fill this gap. Using results in this direction, one can derive various sufficient conditions for the oscillation of Eq. (2). According to Remark 3.1, neither of them can be optimal in the above sense but, nevertheless, they are of interest since they cannot be derived from other known results in the literature. We combine Theorem 3.1 with the result ([19], Corollary 1) to obtain

Theorem 3.4 ([27]) Let K and k be defined by $(C_2)''$,$(C_3)''$, $0 \le k \le 1/e$ and

$$K > k + \frac{1}{\lambda(k)} - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2} \qquad (C_{11})'$$

where $\lambda(k)$ is the smaller root of the equation $\lambda = \exp(k\lambda)$. Then Eq. (2) is oscillatory.

Note that the condition $(C_{11})'$ is analogous to the condition (C_{11}).

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A STEENROD-MILNOR ACTION ORDERING ON DICKSON INVARIANTS

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ABSTRACT. Let $f : (E(x_1, ..., x_k) \otimes P[y_1, ..., y_k])^{GL_k} \to (E(x_1, ..., x_k) \otimes P[y_1, ..., y_k])^{GL_k}$ be a degree preserving Steenrod module map such that f is an isomorphism on degree $2p^{k-1}(p-1)$. Using a particular ordering depending on the dual Milnor basis we show that f is an upper triangular map, hence an isomorphism.

1. INTRODUCTION

Motivated by topological questions regarding the cohomology of an infinite (finite) loop space and influenced by the work of Campbell, Cohen, Peterson and Selick in [1] and [2] we study the problem under which conditions is a Steenrod module map between the full rings of invariants of $GL(k, \mathbb{Z}/p\mathbb{Z})$ an isomorphism. In a sequel we study the same problem between certain quotients of the full ring of invariants [4]. It turns out that although the same result holds its proof is more technical.

It is known that given a monomial d^n there exists a unique p-th power Steenrod operation P^{p^m} of smallest degree such that $P^{p^m} d^n \neq 0$. Thus there exists a set consisting of p-th powers of generators $d_{k,i}^{p^{t_i}}$ such that $d_{k,i}^{p^{t_i}} \backslash d^n$ and $t_i + i - 1 = m$. It is obvious that $P^{p^{t_i}} ... P^{p^m} d^n \neq 0$. We are interested in finding the longest such sequence of Steenrod operations. Of course it depends on m and i. The required sequence shares the property that $P^{p^{t_{i(l)}}} ... P^{p^m} d^n$ is also a monomial according to proposition 1 e). We call such a sequence a **Steenrod-Milnor** action on d^n. Now we iterate this procedure on the monomial $P^{p^{t_i}} ... P^{p^m} d^n$ until the resulting monomial is $d_{k,0}^{p^q}$ for the smallest p^q.

Theorem 5 *There exists a sequence of Steenrod-Milnor operations P^{Γ} such that $P^{\Gamma} d^n = \lambda d_{n,0}^{p^{l(n)}}$. Here $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$.*

Next, given two monomials d^n and $d^{n'}$ we define an ordering according to their first different Steenrod-Milnor actions $P^{p^{t_{i(l)}}} ... P^{p^m}$ and $P^{p^{t'_{i(l')}}} ... P^{p^{m'}}$. We call this action a Steenrod-Milnor action ordering. Using this action we prove the following Theorem:

Theorem 6 *Let $f : (E(x_1, ..., x_k) \otimes P[y_1, ..., y_k])^{GL_k} \to (E(x_1, ..., x_k) \otimes P[y_1, ..., y_k])^{GL_k}$ be a Steenrod module map which preserves the degree such that $f(d_{k,k-1}) = \lambda d_{k,k-1}$ for $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$. Then f is a lower triangular map with respect to S-M ordering and hence an isomorphism.*

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A consequence of this result is the well known Theorem of Campbell, Peterson and Selick:

Theorem [1]*Let* $f : \Omega_0^\infty S^\infty \to \Omega_0^\infty S^\infty$ be an H-map which induces an isomorphism on $H_{2p-3}(\Omega_0^\infty S^\infty; \mathbb{Z}/p\mathbb{Z})$. If $p > 2$ suppose in addition that f is a loop map or that

$$f_*(d_{2,0})^* = \lambda(d_{2,0})^*$$

for some $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$. Then $f_{(p)}$ is a homotopy equivalence. *Here* $(d_{2,0})^*$ is the hom-dual of the top degree Dickson generator in D_2.

2. A Steenrod-Milnor action ordering on Dickson invariants

We shall recall some well known Theorems concerning the action of the Steenrod algebra on Dickson algebra generators. Let us also recall the full ring of invariants of $GL(k, \mathbb{Z}/p\mathbb{Z})$.

Let E_k stand for $E(x_1, ..., x_k)$ and S_k for $P[y_1, ..., y_k]$. Here $|x_i| = 1$ and $|y_i| = 2$ with $\beta x_i = y_i$.

Theorem 1. *The Dickson algebra* ${S_k}^{GL_k}$ *is a polynomial algebra on* $\{d_{k,0}, ..., d_{k,k-1}\}$.

The Dickson algebra generators are defined bellow.

Theorem 2. [5] *The algebra* $(E_k \otimes S_k)^{GL_k}$ *is a tensor product between the polynomial algebra* D_k *and the* $\mathbb{Z}/p\mathbb{Z}$*-module spanned by the set of elements consisting of the following monomials:*

$$M_{k;s_1,...,s_l} L_k^{p-2};\ 0 \le l \le k-1,\ \textit{and}\ 0 \le s_1 < \cdots < s_l \le k-1.$$

Here $l = 0$ *implies that* $M_k = x_1 ... x_k$*. Its algebra structure is determined by the following relations:*

a) $(M_{k;s_1,...,s_l} L_k^{p-2})^2 = 0$ *for* $0 \le l \le k-1$*, and* $0 \le s_1 < \cdots < s_l \le k-1$.

b) $M_{k;s_1,...,s_l} L_k^{(p-2)} d_{k,k-1}^{m-1} = (-1)^{(k-l)(k-l-1)/2} \prod_{t=1}^{k-l} M_{k;0,...,\widehat{k-s_t},...,k-1} L_k^{p-2}$.

Here $0 \le l \le k-1$*, and* $0 \le s_1 < \cdots < s_l \le k-1$.

The elements above have been defined by Mui in [5] as follows:

$$M_{k;s_1,...,s_l} = \frac{1}{(k-l)!} \begin{vmatrix} x_1 & \cdots & x_1 \\ \vdots & & \vdots \\ x_1 & \cdots & x_k \\ y_1^{p^{s_1}} & \cdots & y_k^{p^{s_1}} \\ \vdots & & \vdots \\ y_1^{p^{s_l}} & \cdots & y_k^{p^{s_l}} \end{vmatrix} \quad d_{k,i} = \frac{L_{k,i}}{L_k} \quad L_{k,i} = \begin{vmatrix} y_1 & \cdots & x_k \\ y_1^p & \cdots & y_k^p \\ \vdots & & \vdots \\ y_1^{p^k} & \cdots & y_k^{p^k} \end{vmatrix}$$

Here there are $k-l$ rows of x_i's and the s_i-th's powers are completing the rest of the first determinant, where $0 \le s_1 < \cdots < s_l \le k-1$. The row $y_1^{p^i} ... y_k^{p^i}$ is omitted in the second determinant. $L_k := L_{k,k}$.

$|M_{k;s_1,...,s_l}| = k - l + 2(p^{s_1} + \cdots + p^{s_l})$ and $|L_{k,i}| = 2(1 + \cdots + p^k - p^i)$.

Theorem 3. [3] *1)* $P^{p^j}(d_{k,i}) = \begin{cases} d_{k,i-1}, \text{ if } j = i-1 \\ -d_{k,i}d_{k,k-1}, \text{ if } j = k-1 \\ 0, \text{ otherwise} \end{cases}$

2) $P^{p^j}(M_{k;s_1,\dots,s_l}L_k^{p-2}) = \begin{cases} M_{k;s_1,\dots,s_t+1,\dots,s_l}L_k^{p-2};\ j = s_t,\ s_{t+1} \neq s_t+1 \\ (p-2)M_{k;s_1,\dots,s_l}L_k^{p-2}d_{k,k-1};\ j = k-1,\ s_l \neq k-1 \\ -L_k^{p-2}(M_{k;s_1,\dots,s_l}d_{k,k-1} + \sum\limits_{s'_t \notin \{s_1,\dots,s_l\}} (-1)^t M_{k;s_1,\dots,s'_t,\dots,s_l}d_{k,s_t}); \\ j = k-1,\ s_l = k-1 \\ 0;\ j = s_t - 1 = s_{t-1},\ l-1 = s_t, k \end{cases}$

Lemma 1. $P^{p^t}(d_{k,i}^{p^l}) = \begin{cases} d_{k,i-1}^{p^l}, \text{ if } t = l+i-1 \\ -d_{k,i}^{p^l}d_{k,i-1}^{p^l}, \text{ if } t = l+k-1 \\ 0, \text{ otherwise} \end{cases}$.

Theorem 4. [3] *1) Let $q > 0$. If $q = \sum\limits_i^{k-1} a_t p^{t+l}$ such that $p-1 \geq a_t \geq a_{t-1} > a_{i-1} = 0$. Then*

$$P^q d_{k,0}^{p^l} = d_{k,0}^{p^l}(-1)^{a_{k-1}} \prod_i^{k-1} \binom{a_t}{a_{t-1}} d_{k,t}^{p^l(a_t - a_{t-1})}$$

Otherwise, $P^q d_{k,0}^{p^l} = 0$.

2) Let $q = \sum\limits_s^{k-1} a_t p^{t+l} > 0$ such that $p-1 \geq a_t \geq a_{t-1} \geq a_i \geq 0$ and $a_i + 1 \geq a_{i-1} \geq a_t \geq a_{t-1} \geq a_{s-1} = 0$. Then

$$P^q d_{k,i}^{p^l} = d_{k,i}^{p^l}(-1)^{a_{k-1}} \left(\prod_{i+1}^{k-1} \binom{a_t}{a_{t-1}}\right) \binom{a_i+1}{a_{i-1}} \left(\prod_s^{i-1} \binom{a_t}{a_{t-1}}\right) \prod_s^{k-1} d_{k,t}^{p^l(a_t - a_{t-1})}$$

Here $a_{s-1} = 0$. Otherwise, $P^q d_{k,0}^{p^l} = 0$.

Remark 1. *Please note that the case $a_i = 0$ and $a_{i-1} = 1$ is allowed in the Theorem above.*

We shall apply formulas above on a Dickson algebra monomial starting with the lower non-zero p-th power.

Definition 1. *Let $n = (n_0, \dots, n_{k-1})$ be a sequence of non-negative integers and $d^n = \prod_i d_{k,i}^{n_i}$ a monomial in the Dickson algebra. Let $n_i = \sum\limits_{t=0}^{l(i)} a_{i,t}p^{n_{i,t}}$ be the n_i's p-adic expansion with $\prod a_{i,t} \neq 0$. a) Let $M := \{m_0, m_1, \dots, m_{l(n)} \mid m_i < m_{i+1}\} = \{n_{0,t} + k - 1, n_{i,s} + i - 1 \mid 0 \leq t \leq l(0),\ 1 \leq i \leq k-1 \text{ and } 0 \leq s \leq l(i)\}$.*
b) Let $I(m_j, n) := (i_1, \dots, i_r)$ such that $m_j = n_{0,t} + k - 1 = n_{i_r,s} + i_r - 1$ and $i_{I(m_j,n)} := \begin{cases} \max I(m_j, n), \text{ if } 0 \notin I(m_j, n) \\ k, \text{ if } 0 \in I(m_j, n) \end{cases}$.

c) Let $P^{\Gamma(m,l)}$ stand for the Steenrod operation $P^{p^{m-l+1}}P^{p^{m-l+2}}\dots P^{p^m}$. Let us call $P^{\Gamma(m,l)}$ a ***Steenrod-Milnor action*** *of type (m, l).*

Proposition 1. *a)* $P^{\Gamma(m_0,k)}d_{k,0}^{p^{n_0}} = -d_{k,0}^{2p^{n_0}}$. *Here $m_0 = n_0 + k - 1$.*

b) $\underbrace{P^{\Gamma(m_0,k)}\dots P^{\Gamma(m_0,k)}}_{p-1} = -d_{k,0}^{p^{n_0+1}}$. *Here $m_0 = n_0 + k - 1$.*

c) Let $d^n \in D_k$ *and* $m_0 \in M$, *then*
$$P^{p^{m_0}} d^n = \sum_{0<i_r \in I(m_0,d^n)} a_{i_r,0} d^n d_{k,i_r-1}^{p^{n_{i_r,0}}} d_{k,i_r}^{-p^{n_{i_r,0}}} + d^n d_{k,0}^{a_{0,0}p^{n_{0,0}}} P^{p^{m_0}} d_{k,0}^{a_{0,0}p^{n_{0,0}}}.$$

d) Let $d^n \in D_k$ *and* $m_0 \in M$, *then*
$$P^{\Gamma(m_0, i_{I(m_0,n)})} d^n = \begin{cases} a_{i_{I(m_0,n)},0} d^n d_{k,0}^{p^{n_{i_{I(m_0,n)},0}}} d_{k,i_{I(m_0,n)}}^{-p^{n_{i_{I(m_0,n)},0}}}, & \text{if } 0 \notin I(m_0, d^n) \\ -a_{0,0} d^n d_{k,0}^{p^{n_{0,0}}}, & \text{if } 0 \in I(m_0, d^n) \end{cases}.$$

e) Let $d^n \in D_k$ *and* $m_0 \in M$, *then* $\underbrace{P^{\Gamma(m_0, i_{I(m_0,n)})} \ldots P^{\Gamma(m_0, i_{I(m_0,n)})}}_{a_{i_{I(m_0,n)},0}} d^n =$
$$\left(a_{i_{I(m_0,n)},0}\right)! d^n d_{k,0}^{a_{i_{I(m_0,n)},0} p^{n_{i_{I(m_0,n)},0}}} d_{k,i_{I(m_0,n)}}^{-a_{i_{I(m_0,n)},0} p^{n_{i_{I(m_0,n)},0}}} \quad \text{or}$$
$$\underbrace{P^{\Gamma(m_0,k)} \ldots P^{\Gamma(m_0,k)}}_{p-1-a_{0,0}} d^n = (-1)^{p-1-a_{0,0}} \frac{(p-1)!}{(a_{0,0}-1)!} d^n d_{k,0}^{p^{n_{0,0}+1}} d_{k,0}^{-a_{0,0}p^{n_{0,0}}}.$$

Proof. a) By lemma 1 $P^{p^t} d_{k,0}^{p^l} = 0$, if $t \neq l+k-1$ and $P^{p^t} d_{k,i}^{p^l} = 0$, if $t \neq l+k-1$ or $l+i-1$. Now the statement follows using Cartan formula.

b) is an application of a).

c) Since $m_0 = \max M$, Theorem 3 and Cartan formula implies the statement.

d) Let $m_0 = n_0 + k - 1 = n_i + i - 1$ for $i > 0$. By lemma 1 $P^{p^{m_0-i}} P^{p^{m_0-i+1}} \ldots P^{p^{m_0}} d_{k,i}^{p^{n_i}} = P^{p^{m_0-i}} d_{k,0}^{p^{n_i}} = P^{p^{n_i-1}} d_{k,0}^{p^{n_i}} = 0$. Now the statement is an application of c).

e) This is a repeated application of d). Two main cases should be considered depending on $i_{I(m_0,n)}$. Moreover, the number of times the S-M operation has to be applied depends on $a_{i_{I(m_0,n)},0}$. We describe the first step in details. The next steps follow the same pattern. Let us compare d^n and $d^n d_{k,0}^{p^{n_{i_{I(m_0,n)},0}}} d_{k,i_{I(m_0,n)}}^{-p^{n_{i_{I(m_0,n)},0}}}$. Let M and M' be the corresponding sets defined in definition 1.
Let $i_{I(m_0,n)} > 0$, then $n_{i_{I(m_0,n)},0} + k - 1 > m_0$. If $a_{i_{I(m_0,n)},0} = 1$ and $I(m_0, n) = \{i_{I(m_0,n)}\}$, then $m'_0 = \min\{m_1, n_{i_{I(m_0,n)},0} + k - 1\} > m_0$. Otherwise, $m'_0 = m_0$.
Let $i_{i_{I(m_0,n)},0} = 0$ and $a_{0,0} < p-1$, then $m'_0 = m_0$. Otherwise, $m'_0 = m_0 + 1$. Now the statement follows. ∎

Let us comment on the statement of last Proposition. Let d^n be a monomial and $d^{n'}$ the resulting monomial as in the statement of e) above. If for each index $i_r \in I(m_0, n)$ a suitable Steenrod-Milnor operation is defined, then the smallest p-th component of exponents of $d_{k,i}$'s are reduced and that of $d_{k,0}$'s is increased respectively.

Corollary 1. *Let* $d^n \in D_k$ *and* $i_{r_t} \in I(m_0, n) = \{i_{r_1}, \ldots, i_{r_l}\}$. *a) If* $0 < i_{r_1}$, *then*
$$\underbrace{P^{\Gamma(m_0,i_{r_1})}}_{a_{i_{r_1},0}} \underbrace{P^{\Gamma(m_0,i_{r_2})}}_{a_{i_{r_2},0}} \ldots \underbrace{P^{\Gamma(m_0,i_{r_l})}}_{a_{i_l,0}} d^n = \lambda d^n d_{k,0}^{(\sum_{0<i_r \in I(m_0,n)} a_{i_r,0} p^{n_{i_t,0}})} \prod_{i_r \in I(m_0,n)} d_{k,i_r}^{-a_{i_r,0} p^{n_{i_r,0}}}.$$
b) If $0 = i_{r_1}$, *then*
$$\underbrace{P^{\Gamma(m_0,i_{r_2})}}_{a_{i_{r_2},0}} \ldots \underbrace{P^{\Gamma(m_0,i_{r_l})}}_{a_{i_{r_l},0}} \underbrace{P^{\Gamma(m_0,k)}}_{p-1-a_{0,0}} d^n = \lambda d^n d_{k,0}^{(p^{k_{0,0}+1} + \sum_{0<i_r \in I(m_0,n)} a_{i_r,0} p^{n_{i_t,0}})} \prod_{i_r \in I(m_0,n)} d_{k,i_r}^{-a_{i_r,0} p^{n_{i_r,0}}}.$$
Here $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$.

Proof. This is an application of Proposition 1 e). ∎

Let d^n be a monomial and $d^{n'}$ the resulting monomial as in last corollary. Let M and M' be as in definition 1, then $m_0 < m_0'$.

Definition 2. *Let m be a positive integer, $I = (i_1, ..., i_l)$ a strictly increasing sequence of integers between 0 and $k-1$, and $J = (a_1, ..., a_l)$ a sequence of integers between 0 and $p-1$. We define $P^{\Gamma(m,I,J)}$ the following S-M operation:*
a) if $i_1 = 0$, $P^{\Gamma(m,I,J)} = \underbrace{P^{\Gamma(m,i_2)}}_{a_2} ... \underbrace{P^{\Gamma(m,i_l)}}_{a_l} \underbrace{P^{\Gamma(m,k)}}_{p-1-a_1}$
b) If $0 < i_1$, $P^{\Gamma(m,I,J)} = \underbrace{P^{\Gamma(m,i_1)}}_{a_1} ... \underbrace{P^{\Gamma(m,i_l)}}_{a_l}$.

Theorem 5. *There exists a sequence of Steenrod-Milnor operations P^Γ such that $P^\Gamma d^n = \lambda d_{n,0}^{p^{l(n)}}$. Here $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$.*

Proof. We shall describe an algorithm which constructs the required sequence. This algorithm depends heavily on last corollary.

Step 0. Let $P^\Gamma = P^0$.

Step 1. Given d^n define $I(m_0, n)$, $J(m_0, n) = (a_{i_1,0}, ..., a_{i_l,0})$ and $i_{r(I(m_0,n))}$ as in Definition 1 b). Define $P^\Gamma := P^{\Gamma(m_0,I,J)} P^\Gamma$.

Step 2. Define $d^n := \lambda d^n d_{k,0}^{(\sum_{0<i_r \in I(m_0,n)} a_{i_r,0} p^{n_{i_t,0}})} \prod_{i_r \in I(m_0,n)} d_{k,i_r}^{-a_{i_r,0} p^{n_{i_r,0}}}$ or

$\lambda d^n d_{k,0}^{(p^{k_{0,0}+1} + \sum_{0<i_r \in I(m_0,n)} a_{i_r,0} p^{n_{i_t,0}})} \prod_{i_r \in I(m_0,n)} d_{k,i_r}^{-a_{i_r,0} p^{n_{i_r,0}}}$ given by corollary above.

If $n_i > 0$ for some $i > 0$ or $n_0 \neq p^{l(n)}$ for some positive integer $l(n)$, then proceed to step 1. Otherwise, the required sequence is P^Γ. ∎

Lemma 2. *Let d^n and $d^{n'}$ be monomials and $\{M, I(m_0, n), J(m_0, n)\}$, $\{M', I(m_0', n'), J(m_0', n')\}$ their corresponding sequences.*
a) If $m_0 = m_0'$, $I(m_0, n) = I'(m_0, n')$, and $J(m_0, n) = J(m_0, n')$, then $P^{\Gamma(m_0,I,J)}(d^n - d^{n'}) = 0$.
b) If $m_0 = m_0'$, $I(m_0, n) = I'(m_0, n')$, and $\exists t_0 > 0$ such that $a_{i_{t_0},0} > a'_{i_{t_0},0}$, then $P^{\Gamma(m_0,I,J)}(d^n - d^{n'}) = P^{\Gamma(m_0,I,J)}(d^n)$.
c) If $m_0 = m_0'$, $I(m_0, n) = I'(m_0, n')$, and $0 < a_{0,0} < a'_{0,0}$, then $P^{\Gamma(m_0,I,J)}(d^n - d^{n'}) = P^{\Gamma(m_0,I,J)}(d^n)$.
d) If $m_0 = m_0'$ and either $0 \notin I(m_0, n) \cap I(m_0, n')$ or $a_{0,0} = a'_{0,0}$, then $P^{\Gamma(m_0,I,J)}(d^n - d^{n'}) = P^{\Gamma(m_0,I,J)}(d^n)$.
e) If $m_0 = m_0'$ and either $0 < a_{0,0} < a'_{0,0}$ or $0 = a'_{0,0} < a_{0,0}$, then $P^{\Gamma(m_0,I,J)}(d^n - d^{n'}) = P^{\Gamma(m_0,I,J)}(d^n)$.
f) If $m_0 < m_0'$, then $P^{\Gamma(m_0,I,J)}(d^n - d^{n'}) = P^{\Gamma(m_0,I,J)}(d^n)$.

P^Γ as in the last Theorem is a repeated S-M action. Applying lemma above we define an ordering in D_k using the corresponding action and call it a **Steenrod-Milnor action ordering** and write S-M ordering.

Definition 3. *Let d^n, $d^{n'} \in D_k$ and $n_i = \sum_{t=0}^{l(i)} a_{i,t} p^{n_{i,t}}$, $n_i = \sum_{t=0}^{l'(i)} a'_{i,t} p^{n'_{i,t}}$. Here $\prod_{i,t} a_{i,t} \prod_{i,t} a'_{i,t} \neq 0$. 1) i) If $m_0 < m_0'$, we call $d^n < d^{n'}$.*

ii) If $m_0 = m_0'$ and one of hypotheses of last lemma is applied, we call $d^n < d^{n'}$.

iii) If $m_0 = m'_0$ and none of hypotheses of last lemma is applied, then the ordering is defined according to monomials $P^{\Gamma(m_0,I,J)}d^n$ and $P^{\Gamma(m_0,I,J)}d^{n'}$. Here $J = (a_{i_1,0}, ..., a_{i_l,0})$.

Next we extend the previous ideas to exterior monomials.

Lemma 3. *1) Let $M_{k;s_1,...,s_l}L_k^{p-2}d^n$ be a monomial in $(E(x_1,...,x_k)\otimes P[y_1,...,y_k])^{GL_k}$ and $P^B := \beta\underbrace{P^{p^0}\beta}\dots\underbrace{P^{p^{k-l-2}}...P^{p^0}\beta}\underbrace{P^{p^{k-l-1}}...P^{p^{s_1}}}\dots\underbrace{P^{p^{k-2}}...P^{p^{s_l}}}$. Then*

$$P^B M_{k;s_1,...,s_l}L_k^{p-2}d^n = (-1)^{(k-l-1)!}d_{k,0}d^n$$

If $s_l = k-1$, then the result follows applying
$P^B := \beta\underbrace{P^{p^0}\beta}\dots\underbrace{P^{p^{k-l-2}}...P^{p^0}\beta}\underbrace{P^{p^{k-l-1}}...P^{p^{s_1}}}\dots\underbrace{P^{p^{k-3}}...P^{p^{s_l-1}}}$.

2) Let $M_{k;s_1,...,s_l}L_k^{p-2}d^n$ and $M_{k;s'_1,...,s'_{l'}}L_k^{p-2}d^{n'}$ be monomials such that $s_{l-t} < s'_{l'-t}$ and t is minimal with this property, then $P^B M_{k;s'_1,...,s'_{l'}}L_k^{p-2}d^{n'} = 0$. Here P^B is as in 1).

Proof. Let us recall that $P^{p^{s_l}}(M_{k;s_1,...,s_l}L_k^{p-2}) = M_{k;s_1,...,s_l-1,s_l+1}L_k^{p-2}$ for $s_l < k-1$ and $P^{p^{s_l}}d_{k;t}^{a_tp^{n_t}} \neq 0$ if and only if $n_t = s_l - t + 1$ for $0 \le t \le s_l + 1$. If $0 = s_l$, we apply the Bockstein operation β. Thus $P^{p^{k-2}}...P^{p^{s_l}}M_{k;s_1,...,s_l}L_k^{p-2}d^n = \sum_{0}^{k-1-s_l} M_{k;s_1,...,s_{l-1},s_l+t_l}L_k^{p-2}f_{t_l}$. Here f_{t_l} is a polynomial in D_k.

Let $P^E = \underbrace{P^{p^{k-l-1}}...P^{p^{s_1}}}\dots\underbrace{P^{p^{k-2}}...P^{p^{s_l}}}$. Iterating the last formula we obtain:

$$P^E M_{k;s_1,...,s_l}L_k^{p-2}d^n = \sum_{q=1}^{l}\sum_{0}^{s_{q+1}+t_{q+1}-s_q} M_{k;s_1+t_1,...,s_{l-1}+t_{l-1},s_l+t_l}L_k^{p-2}f_{t_1,...,t_l}$$

Here $s_{l+1} = 0$ and $t_{l+1} = k-1$.
Let us suppose that $s_1 + t_1 < k - l$. Let $P^\Delta = P^{p^{k-l-2}}...P^{p^0}\beta$ and $A = M_{k;s_1+t_1,...,s_{l-1}+t_{l-1},s_l+t_l}L_k^{p-2}f_{t_1,...,t_l}$. There are $s_1+t_1-1 \le k-l-2$ positions to be filled by powers of y's using Steenrod operations: $\beta\underbrace{P^{p^0}\beta}\dots\underbrace{P^{p^{k-l-2}}...P^{p^0}\beta}$. Since there are $k-l$ β's in this sequence and only $s_1+t_1-1 \le k-l-2$ positions, it is obvious that $P^\Delta A = 0$. Now suppose that $s_1 + t_1 = k - l$ and one operation P^{p^q} of P^Δ is not applied on A. Then it will be less positions than the number of remaining β's. In that case $\beta\underbrace{P^{p^0}\beta}\dots\underbrace{P^{p^{k-l-2}}...P^{p^0}\beta}M_{k;s_1+t_1,...,s_{l-1}+t_{l-1},s_l+t_l}L_k^{p-2}f_{t_1,...,t_l} = 0$.
The claim follows. ∎

Definition 4. *3) i) $M_{k;s_1,...,s_l}L_k^{p-2}d^n < M_{k;s_1,...,s_l}L_k^{p-2}d^{n'}$, if $d_{k,0}d^n < d_{k,0}d^{n'}$.*
ii) $M_{k;s_1,...,s_l}L_k^{p-2}d^n < M_{k;s'_1,...,s'_{l'}}L_k^{p-2}d^{n'}$, if $s_t < s'_t$ and t is maximal with this property.

Remark 2. *Because of our definitions, the S-M action ordering is a total ordering.*

Corollary 2. *Let $M_{k;s_1,...,s_l}L_k^{p-2}d^n \in (E_k\otimes S_k)^{GL_k}$. There exists a sequence of S-M operations $P^{\Gamma(M_{k;s_1,...,s_l}L_k^{p-2}d^n)}$ such that*

$$P^{\Gamma(d_{k,0}d^n)}\beta P^{p^0}\beta\dots\underbrace{P^{p^{k-l-2}}...P^{p^0}\beta}\underbrace{P^{p^{k-l-1}}...P^{p^{s_1}}}\dots\underbrace{P^{p^{k-2}}...P^{p^{s_l}}}M_{k;s_1,...,s_l}L_k^{p-2}d^n = \lambda d_{k,0}^{p^q}$$

and q is minimal with this property.
Here $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$.

Now we are ready to proceed to our main Theorem.

Theorem 6. *Let* $f : (E_k \otimes S_k)^{GL_k} \to (E_k \otimes S_k)^{GL_k}$ *be a Steenrod module map which preserves the degree such that* $f(d_{k,k-1}) = \lambda d_{k,k-1}$ *for* $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$. *Then f is a lower triangular map with respect to S-M ordering and hence an isomorphism.*

Proof. By hypothesis and Theorem 3, $f(d_{k,i}) = \lambda d_{k,i}$ for $i = 0, ..., k-1$ after applying a suitable Steenrod operation.

Let $d^n \in D_k$ and $(d^{n(1)}, ..., d^{n(l(|d^n|))})$ the increasing sequence of elements of degree $|d^n|$. Let $f(d^n) = \sum_{t=1}^{l|d^n|} a_t d^{n(t)}$. Claim: If $d^{n(t_0)} = d^n$, then $a_t \equiv 0 \bmod p$ for $t < t_0$. We use induction on t for $t < t_0$. $P^{\Gamma(d^{n(1)})} f(d^n) = P^{\Gamma(d^{n(1)})} \sum_{t=1}^{l|d^n|} a_t d^{n(t)}$ implies $a_1 \equiv 0 \bmod p$. $P^{\Gamma(d^{n(i)})} f(d^n) = P^{\Gamma(d^{n(i)})} \sum_{t=i}^{l|d^n|} a_t d^{n(t)}$ implies $a_i \equiv 0 \bmod p$ for $i < t_0$. Now using Proposition 1 and the fact $f(d_{k,0}) = \lambda d_{k,0}$ for $\lambda \not\equiv 0 \bmod p$, we conclude that $a_{t_0} \not\equiv 0 \bmod p$. Hence f is a lower triangular map.

Because of the direct sum decomposition of the ring of invariance, it follows that $f(M_{k;s_1,...,s_l} L_k^{p-2} d^n) = \alpha M_{k;s_1,...,s_l} L_k^{p-2} d$, then
$\beta P^{p^0} \beta ... \underbrace{P^{p^{k-l-2}} ... P^{p^0}}\beta \underbrace{P^{p^{k-l-1}} ... P^{p^{s_1}}} ... \underbrace{P^{p^{k-2}} ... P^{p^{s_l}}} f(M_{k;s_1,...,s_l} L_k^{p-2} d^n) = \lambda d_{k,0} d$
and the claim follows. ∎

Remark 3. *Please note that for* $k = 1$ *it suffices to require* $f(M_{1;1} L_1^{p-2}) = \lambda(M_{1;1} L_1^{p-2})$, *since* $\beta(M_{1;1} L_1^{p-2}) = d_{1,0}$.

Corollary 3. *a) Let* $S(E_k \otimes S_k)^{GL_k}$ *be the subalgebra of* $(E_k \otimes S_k)^{GL_k}$ *generated by*
$\{d_{k,i}, M_{k,s_1,...,s_{k-1}}, M_{k,s'_1,...,s'_{k-3},k-1}\}$ *where* $0 \le i \le k-1$, $0 \le s_1 < ... < s_{k-1} \le k-1$ *and* $0 \le s'_1 < ... < s'_{k-3} \le k-2$. *If* $f : S(E_k \otimes S_k)^{GL_k} \to S(E_k \otimes S_k)^{GL_k}$ *satisfies* $f(d_{k,k-1}) = \lambda d_{k,k-1}$, *then f is an isomorphism.*

b) Let $I[k]$ *be the ideal of* $S(E_k \otimes S_k)^{GL_k}$ *generated by* $\{d_{k,0}, M_{k,s_1,...,s_{k-1}}, M_{k,s'_1,...,s'_{k-3},k-1}\}$, *then the induced map f which satisfies* $f(d_{k,0}) = \lambda d_{k,0}$ *is also an isomorphism.*

Corollary b) above is a reformulation of Theorem 4.1 in [1]. We close this work by applying last corollary in the mod $-p$ homology of QS^0.

Let $R =< Q^{(I,J)} | I = (i_1, ..., i_n), J = (\varepsilon_1, ..., \varepsilon_n) >$ be the Dyer-Lashof algebra, then $H_*(Q_0 S^0; \mathbb{Z}/p\mathbb{Z})$ is the free commutative algebra generated by $\Phi(R)$ subject to the following relation $Q^{(I,J)} \approx (Q^{(I',J')})^p$ if $I = (i_1, I')$, $J = (0, J')$ and $exc(Q^{(I,J)}) = 0$. Here $\Phi : R \to H_*(Q_0 S^0; \mathbb{Z}/p\mathbb{Z})$ is the A_*-module map given by $\Phi(Q^{(I,J)}) = Q^{(I,J)}[1] * [-p^{l(I)}]$, $[1]$ is a generator of $\tilde{H}_0(S^0; \mathbb{Z}/p\mathbb{Z})$, $[r] = [1]^r$ and $l(I)$ is the length of I. Thus there exists an A_*-module isomorphism between the generators of $H_*(Q_0 S^0; \mathbb{Z}/p\mathbb{Z})$ and the quotient $R/Q_0 R$ where $Q_0 R = \{Q^{(I,J)} | exc(I, J) = 0\}$. It is known that $R[k]^* \cong S(E_k \otimes S_k)^{GL_k}$ as Steenrod algebras and $(R/Q_0 R)[k]^* \cong I[k]$ as Steenrod modules. Here $R = \bigoplus R[k]$. Now the following Theorem is a consequence of last corollary.

Theorem 7. [1]*Let* $f : \Omega_0^\infty S^\infty \to \Omega_0^\infty S^\infty$ *be an H-map which induces an isomorphism on* $H_{2p-3}(\Omega_0^\infty S^\infty; \mathbb{Z}/p\mathbb{Z})$. *If* $p > 2$ *suppose in addition that f is a loop map* *or that*

$$f_*(d_{2,0})^* = \lambda(d_{2,0})^*$$

for some $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$. *Then* $f_{(p)}$ *is a homotopy equivalence. Here* $(d_{2,0})^*$ *is the* hom-*dual of the top degree Dickson generator in* D_2.

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